Research Statement: Tony Bedenikovic

The statement below is intended to be a non-technical description of my research and my research goals. Intended for a general audience, it gives a sense, I believe, of the type of mathematics that I do.

My area of specialty is low-dimensional topology. In the broadest sense, topologists study collections of points and the effects of transformations on these collections. Often the points in a collection are neatly organized in space, in which case the topologist's work becomes highly visual. For example, a circle drawn on this flat, 2-dimensional page, viewed as the collection of points which lie on it, is a candidate for study. Likewise, a spherical membrane living in 3-dimensional space, perhaps living in the space above this page, is a candidate for study and, in fact, plays a central role in my research. The transformations which are applied to such collections of points are operations which alter the collections in reasonable ways. One transformation deemed to be reasonable is the act of stretching or deforming an object without tearing it. The points of a circle, for example, can flow naturally onto the points of a square without tearing the circle and thus a circle and a square are considered to be topologically the same. Similarly, a spherical membrane in 3-dimensional space is considered to be topologically the same as a cubical membrane and the same as the thin surface of the earth, with its peaks and valleys.

Continuous, organic deformation is not the only transformation allowed on a collection of points. Another transformation deemed to be reasonable is the act of reconfiguring points in a collection such that neighboring points remain neighbors. This transformation moves points to perhaps different points in space but preserves intrinsic properties of the collection. For example, a knotted circuit in 3-dimensional space may be viewed as a reconfiguration of the points on a circle. Imagine tying the laces on tennis shoes and then joining the ends of the laces to form a knotted circuit. A one-to-one correspondence exists between the points of a circle and the points of the knotted circuit and the reconfiguration preserves the sense of relative position. It is said that the knot is an embedding of the circle in 3-dimensional space. The study of knots (i.e., embedded circles in 3-dimensional space) is a classic area of topology which leads, indirectly, to my own area of research.

In knot theory, one investigates the complements of knots in order to understand better the knots themselves. That is, one studies the negative spaces created by knots, as a photographer might study the negative spaces in photographs. With this focus, topologists ask, for example: When are two apparently distinct negative spaces in fact topologically the same? What types of distinct closed paths exist in one of these negative spaces? Is every spherical membrane in one of these negative spaces homotopically trivial? This last question asks whether every spherical membrane in a knot complement can shrink to diameter zero by flowing naturally through the points of the space. These questions, particularly the last, motivated the studies of many topologists in the past century. Elegant and accessible, the mathematics developed to answer them is a great source of pride for topologists.

I myself am not a knot theorist. However, my studies have the flavor of knot theory and, in some sense, generalize it. Imagine again a circle drawn on this page and consider the points which lie on the circle or inside it. This collection of points is known as a 2-dimensional disk. I study embeddings of the 2-dimensional disk in 4-dimensional space. Notice that the dimensions of the objects have increased by one. The source of the embedding is a 2-dimensional disk (rather than a 1-dimensional circle) and the home space is 4-dimensional (rather than 3-dimensional). As with knots, one investigates the complements of these embeddings, which is to say that one investigates the negative spaces created by them. The negative spaces in this new setting, it turns out, are remarkably similar to the negative spaces created by knots, even though their dimensions differ. The similarities are so striking, in fact, that it is unavoidable to ask the same questions of these new spaces. Adding a dimension adds subtlety to the problem, however, and the 3-dimensional techniques of knot theory do not apply directly to this higher-dimensional problem. Consequently, many interesting and natural questions regarding disk complements in 4-dimensional space remain open.

Foremost in my own research is the reminiscent question: Is every spherical membrane in a 4-dimensional disk complement homotopically trivial? [1] Because such a space is difficult to grasp, it is beneficial to be able to view it in different, but equivalent, ways. The idea is that the points of an original collection may be made to flow organically into more comprehensible states while preserving the interesting properties of the original state. The main result in [B1] provides an alternative, 3-dimensional description for a vast collection of spaces. [2] Knot complements in 3-dimensional space play a significant role in this new description. Although 3-dimensional states are comfortable for topologists, their applications to 4-dimensional problems have not been fully explored. One of my goals for future research is to apply 3-dimensional techniques to 4-dimensional problems to obtain positive results.

A success in this direction is achieved in [B3]. The 3-dimensional states of disk complements are exploited to obtain a condition which implies the triviality of spherical membranes. The interaction between topology and algebra is an appealing aspect of the subject and in [B3] the topological issue of spherical membranes in a 4-dimensional negative space is translated into an entirely algebraic issue. [3] Furthermore, examples indicate that the algebraic translation is necessary and sufficient for a sizable subset of the problem. [4] Another goal for future research is to understand better the size of this subset for which the algebraic characterization is necessary and sufficient. Disk complements outside this subset appear to be topologically the same as knot complements and therefore the grand results of knot theory apply to them. If the preliminary evidence holds and these complements are indeed knot complements, then the problem reduces to those disks which are in the subset, meaning those disks which possess a special characterizing property. A natural focus for future study, therefore, is this special property and its implications on the triviality of spherical membranes.

Endnotes

1. Note that additional conditions must be imposed on the disk, since arbitrary disk complements in the 4-ball are known to be arbitrarily complicated. I consider disks whose projections onto the boundary of the 4-ball are ribbons, meaning that they are singular disks in 3-space whose self-intersections are of a particular type. For more details about ribbons, see [B3], which is included in this tenure packet. Ribbon disk complements, like knot complements, have 2-dimensional spines which are subcomplexes of contractible 2-complexes. Consequently, ribbons play the role today that knots played a half-century earlier in the exploration of the Whitehead conjecture that every subcomplex of an aspherical 2-complex is itself aspherical.

2. In particular, it is shown in [B1] that every finite, 2-dimensional CW complex may be viewed, up to 3-deformation, as a graph complement in a cube with handles, together with the cone over the boundary of the cube with handles. This 3-dimensional description is a singular 3-manifold with one singular point, the cone point.

3. Using the techniques of [B1], a ribbon disk complement may be viewed as a graph complement in the 3-sphere together with a collection of 2-dimensional disks attached abstractly to the graph complement. These disks are attached along the meridians of a cube with handles which is a regular neighborhood of the embedded disk. Let Y denote the ribbon disk complement, as described above, and let X denote the subspace of Y which consists of the graph complement. It is shown in [B3] that if the second relative homotopy group $\pi_2(Y,X)$ is a free group, then Y is aspherical. (It is known, by the way, that the second relative homotopy group is free in the category of crossed modules.)

4. To each ribbon disk a ribbon graph is associated. The ribbon graph contains the original ribbon knot and differs from it by a collection of spanning arcs, one spanning arc for each ribbon singularity. If this ribbon graph is unknotted in 3-space, meaning that the fundamental group of its complement is a free group, then the ribbon disk complement Y is aspherical if and only if $\pi_2(Y,X)$ is a free group.

Bibliography

[B1] T. Bedenikovic, "Two-complexes as graph complement cone complexes," Topology Proceedings, **27** (2003) no. 1, pp. 27 – 38.

[B2] ____________ , "A class of 3-complexes with infinite cyclic fundamental group," Journal of Knot Theory and Its Ramifications, **13** (2004) no. 4, pp. 565 – 570.

[B3] ____________ , " A sufficient condition for the asphericity of a ribbon disk complement," in preparation.