A solution to POTW 255 *

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PROBLEM. Ten couples attend a Christmas gift exchange. Each person brings a small gift. The gifts are selected at random by drawing names out of a hat. If anyone draws their own name or that of their spouse, all names are replaced and a new drawing is held. How many permissible drawings are there? What is the probability that the first drawing is permissible?

SOLUTION. This problem is an example of counting permutations with forbidden positions. It's also a situation — arising often in mathematics — where a more general problem is easier to solve than a special case. To this end, let's generalize the question and let $N(n)$ denote the number of permissible gift exchanges at a party with n couples where no guest receives their own or their spouse's present. I'll find a general formula for $N(n)$, $N(10)$ being the specific case requested in the problem.

Let's set up some notation. Denote the guests at the party by $1, \ldots, 2n$ with $2k$ and $2k-1$ forming a couple; denote the spouse of k by $\overline{k}$. To start with, let's rephrase this problem in two alternate ways: one algebraic and the second more geometric.

For the first restatement, observe that each drawing describes a permutation, π, of the $2n$ guests: guest k selects the gift brought by $\pi(k)$. A legitimate drawing corresponds to one with the property that $\pi(k) \neq k, \overline{k}$ for $1 \leq k \leq 2n$.

For the second, observe that each drawing describes a $2n \times 2n$ chess board where rooks populate the squares (i, j) whenever guest i selects the gift brought by j. Since a rook attacks any other piece sharing its row or column, any drawing corresponds to the placement of a maximal number of non-attacking rooks, a legitimate drawing corresponds to one in which no rook lies on a square (k, k) or $(k, \overline{k})$ for $1 \leq k \leq 2n$. Let's call this excluded set of squares *forbidden positions.*

For $K \subseteq \{1, \ldots, 2n\}$ let P_K denote the set of permutations of $1, \ldots 2n$ with $\pi(k) \neq k, \overline{k}$ for each $k \in K$ and $\overline{P}_K$ denote its complement in the group of permutations Σ_{2n}. Our aim is to enumerate the set $\overline{P}_1 \cap \cdots \cap \overline{P}_{2n}$. By the *Inclusion-Exclusion Principle* from combinatorics the order of this set is given by the alternating sum

$$\begin{aligned} N(n) = |\overline{P}_1 \cap \cdots \cap \overline{P}_{2n}| &= (2n)! + \sum_{k=1}^{2n} \left((-1)^k \sum_{|K|=k} |P_K| \right) \\ &= (2n)! - \sum_{k} |P_k| + \sum_{j<k} |P_j \cap P_k| + \cdots + |P_1 \cap \cdots \cap P_k| \quad ((1)) \end{aligned}$$

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* http://hilltop.bradley.edu/~delgado/potw/p255.html

Let's interpret the summands. In the geometric language, P_k denotes the boards populated by $2n$ non-attacking rooks which have a rook in square (k,k) or $(k,\overline{k})$, so $|P_k| = 2(2n-1)!$ and $\sum_k |P_k| = 4n(2n-1)!$. In general, we'll have

$$\boxed{\sum_{|K|=k} |P_K| = \left|\begin{matrix}2n\\ k\end{matrix}\right| (2n-k)!,} \qquad ((2))$$

where the symbol $\left|{2n \atop k}\right|$ (read "$2n$ *select* k") denotes the number of ways to place k non-attacking rooks on a $2n \times 2n$ board with each rook sitting on a forbidden position. It is these *selection numbers* that we have to understand. Let $\left\{{2n \atop k}\right\}$ denote the set of such boards.

Some selection numbers are easy to compute, for instance, $\left|{2n \atop 0}\right| = 1$ and $\left|{2n \atop 1}\right| = 4n$; I'll leave it to you to derive $\left|{2n \atop 2n}\right| = 2^n$. Not surprisingly, the selection numbers satisfy a simple recursion relation. Any board in $\left\{{2n-2 \atop k}\right\}$ is, of course, also in $\left\{{2n \atop k}\right\}$; any board in $\left\{{2n-2 \atop k-1}\right\}$ can be extended to one in $\left\{{2n \atop k}\right\}$ in four ways, by placing a rook in one of the positions $(2n-1, 2n-1)$, $(2n-1, 2n)$, $(2n, 2n-1)$ or $(2n, 2n)$; while a board in $\left\{{2n \atop k-2}\right\}$ can be extended to one in $\left\{{2n \atop k}\right\}$ in just two ways, by placing a pair of rooks in positions $\{(2n-1, 2n-1), (2n, 2n)\}$ or $\{(2n-1, 2n), (2n-1, 2n)\}$. Deleting the last two rows and columns from a board in $\left\{{2n \atop k}\right\}$ creates a board either in $\left\{{2n-2 \atop k}\right\}$, $\left\{{2n-2 \atop k-1}\right\}$ or $\left\{{2n-2 \atop k-2}\right\}$. This shows that

$$\boxed{\left|\begin{matrix}2n\\ k\end{matrix}\right| = 2\left|\begin{matrix}2n-2\\ k-2\end{matrix}\right| + 4\left|\begin{matrix}2n-2\\ k-1\end{matrix}\right| + 1\left|\begin{matrix}2n-2\\ k\end{matrix}\right|.} \qquad ((3))$$

The table below shows the values of $\left|{2n \atop k}\right|$, $0 \le k \le 2n$, for $0 \le n \le 6$.

$n = 0:$ 1

$n = 1:$ 1 4 2

$n = 2:$ 1 8 20 16 4

$n = 3:$ 1 12 54 112 108 48 8

$n = 4:$ 1 16 104 352 664 704 416 128 16

$n = 5:$ 1 20 170 800 2280 4064 4560 3200 1360 320 32

$n = 6:$ 1 24 252 1520 5820 14784 25376 29568 23280 12160 4032 768 64

Less obvious is the relation

$$2^{n-k}\left|\begin{matrix}2n\\ k\end{matrix}\right| = \left|\begin{matrix}2n\\ 2n-k\end{matrix}\right| \quad \text{for } 1 \le k \le n. \qquad ((4))$$

You can prove this easily by induction using recursion relation ((3)), however, it also has a nice combinatorial explanation. Suppose you have a board with $2n-k$ rooks in forbidden positions. In the language of the gift exchange, this corresponds to a drawing where $2n-k$

party-goers select forbidden gifts. By the pigeonhole principle, there are at least $n-k$ couples where either the husband and wife swapped gifts or each one received their own gift. Deleting the rooks associated with these couples leaves a board with k rooks in forbidden position. On the other hand, every one of the latter boards can be extended to one with $2n-k$ rooks in forbidden positions in exactly 2^{n-k} ways, which proves the result above.

Using equations ((1)), ((2)) and the recursion relation ((3)) we can compute $N(10)$ and answer the original question. First we find that the left half of the eleventh line of the triangle is

$$1, 40, 740, 8400, 65460, 371328, 1586880, 5218560, 13381920, 26970880, 42904960;$$

you can compute the right half using ((4)). The number of acceptable drawings is therefore

$$\begin{aligned} N(10) = 20! &- 40 \cdot 19! + 740 \cdot 18! - 8400 \cdot 17! + 65460 \cdot 16! - 371328 \cdot 15! + 1586880 \cdot 14! \\ &- 5218560 \cdot 13! + 13381920 \cdot 12! - 26970880 \cdot 11! + 42904960 \cdot 10! \\ &- 53941760 \cdot 9! + 53527680 \cdot 8! - 41748480 \cdot 7! + 25390080 \cdot 6! - 11882496 \cdot 5! \\ &+ 4189440 \cdot 4! - 1075200 \cdot 3! + 189440 \cdot 2! - 20480 \cdot 1! + 1024 \\ &= 312426715251262464 \approx 3 \times 10^{18}. \end{aligned}$$

The probability of an acceptable drawing on the first try is

$$\frac{N(10)}{20!} = \frac{312426715251262464}{2432902008176640000} \approx 0.128417303,$$

or just a bit more than one-eighth.

Having gotten to this point, let's push on and compute the probability of a success on the first drawing for arbitrarily large n. This limiting probability is $\lim_{n\to\infty} N(n)/(2n)!$. A little numerical experimentation gives the following results [*].

n	5	10	15	20	25	30	35	40
$\frac{N(n)}{(2n)!}$	0.121305	0.128417	0.130753	0.131911	0.132602	0.133061	0.133388	0.133633

While the convergence appears to be very slow, a devotee of numbers might recognize a number starting to peek out. In order to move ahead, we need to approximate the value of the selection numbers. The key is the following result.

PROPOSITION. $4^k \binom{n}{k} \le \left|\begin{matrix} 2n \\ k \end{matrix}\right| \le 2^k \binom{2n}{k}$, for $0 \le k \le 2n$.

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[*] The necessary data is located at `http://www.research.att.com/~njas/sequences/b000316.txt`

PROOF. Select k rows of the board. There are two forbidden entries in each row. This provides the upper bound. For the lower bound, select k couples. There are four ways for each couple to select one forbidden gift among them – the husband can select his or his wife's, the wife can select hers or her husband's. This provides a lower bound. ☐

A typical summand in equation ((1)) is of the form $\left|\begin{matrix}2n\\ k\end{matrix}\right|\dfrac{(2n-k)!}{(2n)!}$. From the bound in the proposition, this term is bounded above by

$$2^k\binom{2n}{k}\frac{(2n-k)!}{(2n)!}=2^k\frac{(2n)!}{k!(2n-k)!}\cdot\frac{(2n-k)!}{(2n)!}=\frac{2^k}{k!} \tag*{((5))}$$

and bounded below by

$$\begin{aligned}4^k\binom{n}{k}\frac{(2n-k)!}{(2n)!}&=4^k\frac{n!}{k!(n-k)!}\cdot\frac{(2n-k)!}{(2n)!}\\&=2^k\frac{n(n-1)\cdots(n-k+1)}{k!}\cdot\frac{2^k}{(2n)(2n-1)\cdots(2n-k+1)}\\&=\frac{2^k}{k!}\cdot\frac{(n-1)\cdots(n-(k-1))}{(n-\frac{1}{2})(n-1)\cdots(n-\frac{k-1}{2})}.\end{aligned} \tag*{((6))}$$

The numerator and denominator of this last fraction are both polynomials in n with leading coefficient 1 of the same degree, their ratio therefore goes to 1 as n grows arbitrarily large. The proposition together with ((5)) and ((6)) show:

COROLLARY. $\left|\begin{matrix}2n\\ k\end{matrix}\right|\dfrac{(2n-k)!}{(2n)!}\longrightarrow\dfrac{2^k}{k!}$ as $n\to\infty$. ☐

We now have all the ingredients. From ((1)) and ((2)) we have

$$\frac{N(n)}{(2n)!}=1+\sum_{k=1}^{2n}(-1)^k\left|\begin{matrix}2n\\ k\end{matrix}\right|\frac{(2n-k)!}{(2n)!}.$$

Taking a limit and using the result in the corollary we get

$$\boxed{\lim_{n\to\infty}\frac{N(n)}{(2n)!}=\sum_{k=0}^{\infty}\frac{(-2)^k}{k!}=\frac{1}{e^2}.}$$

Note that $1/e^2\approx 0.135335$ while $N(100)/200!\approx 0.134656$ so the convergence is very slow indeed. Furthermore, since $e^2\approx 7.4$ we would expect about eight drawings before a legitimate drawing comes about.