On Hopf Galois Hirata Extensions

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ABSTRACT. Let $H$ be a finite dimensional Hopf algebra over a field $k$, $H^*$ the dual Hopf algebra of $H$, and $B$ a right $H^*$-Galois and Hirata separable extension of $B^H$. Then $B$ is characterized in terms of the commutator subring $V_B(B^H)$ of $B^H$ in $B$ and the smash product $V_B(B^H)\# H$. A sufficient condition is also given for $B$ to be an $H^*$-Galois Azumaya extension of $B^H$.

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1. Introduction. Let $H$ be a finite dimensional Hopf algebra over a field $k$, $H^*$ the dual Hopf algebra of $H$, and $B$ a right $H^*$-Galois extension of $B^H$. In [3], the class of $H^*$-Galois Azumaya extensions were investigated, and in [8], it was shown that $B$ is a Hirata separable extension of $B^H$ if and only if the commutator subring $V_B(B^H)$ of $B^H$ in $B$ is a left $H$-Galois extension of $C$ where $C$ is the center of $B$ ([8], Lemma 2.1 and Theorem 2.6). The purpose of the present paper is to characterize a right $H^*$-Galois and Hirata separable extension $B$ of $B^H$ in terms of the commutator subring $V_B(B^H)$ and the smash product $V_B(B^H)\# H$. Let $B$ be a right $H^*$-Galois extension of $B^H$. Then the following statements are equivalent:

(1) $B$ is a Hirata separable extension of $B^H$,

(2) $V_B(B^H)$ is an Azumaya $C$-algebra and $V_B(V_B(B^H)) = B^H$,

(3) $V_B(B^H)$ is a right $H^*$-Galois extension of $C$ and a direct summand of $V_B(B^H)\# H$ as a $V_B(B^H)$-bimodule, and
(4) $V_B(B^H)$ is a right $H^*$-Galois extension of $C$, and $V_B(B^H) \# H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$.

Moreover, an equivalent condition is given for a right $H^*$-Galois and Hirata separable extension $B$ of $B^H$ to be an $H^*$-Galois Azumaya extension which was studied in [3] and [7]. Also, let $B$ be a right $H^*$-Galois and Hirata separable extension of $B^H$ and $A$ a subalgebra of $B^H$ over $C$ such that $B^H$ is a projective Hirata separable extension of $A$ containing $A$ as a direct summand as an $A$-bimodule. Then $V_B(A)$ is a separable subalgebra of $B^H$ over $C$, and there exists an $H$-submodule algebra $D$ in $B$ which is separable over $C$ such that $D^H = V_B(A)$ and $D \cong V_B(A) \otimes Z$ as Azumaya $Z$-algebras where $Z$ is the center of $D$ and $F$ is an Azumaya $Z$-algebra in $D$.

2. Basic definitions and notations. Throughout, $H$ denotes a finite dimensional Hopf algebra over a field $k$ with comultiplication $\Delta$ and counit $\varepsilon$, $H^*$ the dual Hopf algebra of $H$, $B$ a left $H$-module algebra, $C$ the center of $B$, $B^H = \{ b \in B \mid hb = \varepsilon(h)b \text{ for all } h \in H \}$ which is called the $H$-invariant of $B$, and $B\#H$ the smash product of $B$ with $H$ where $B\#H = B \otimes_k H$ such that for all $b \# h$ and $b' \# h'$ in $B \# H$, $(b \# h)(b' \# h') = \sum b(h_1 b') \# h_2 h'$ where $\Delta(h) = \sum h_1 \otimes h_2$. $B$ is called a right $H^*$-Galois extension of $B^H$ if $B$ is a right $H^*$-comodule algebra with structure map $\rho : B \rightarrow B \otimes_k H^*$ such that $\beta : B \otimes_{B^H} B \rightarrow B \otimes_k H^*$ is a bijection where $\beta(a \otimes b) = (a \otimes 1)\rho(b)$.

For a subring $A$ of $B$ with the same identity 1, we denote the commutator subring of $A$ in $B$ by $V_B(A)$. we call $B$ a separable extension of $A$ if there exist $\{a_i, b_i\}$ in $B$, $i = 1, 2, ..., m$ for some integer $m$ such that $\sum a_i b_i = 1$, and $\sum a_i \otimes b_i = \sum a_i \otimes b_i b$ for all $b$ in $B$ where $\otimes$ is over $A$. An Azumaya algebra is a separable extension of its center. A ring $B$ is called a Hirata separable extension of $A$ if $B \otimes_A B$ is isomorphic to a direct summand of a finite direct sum of $B$ as a $B$-bimodule. A right $H^*$-Galois extension $B$ is called an $H^*$-Galois Azumaya extension if $B$ is separable over $B^H$ which is an Azumaya algebra over $C^H$. A right $H^*$-Galois extension $B$ of $B^H$ is called an $H^*$-Galois Hirata extension if $B$ is also a Hirata separable extension of $B^H$. Throughout, an $H^*$-Galois
extension means a right $H^*$-Galois extension unless it is stated otherwise.

3. The $H^*$-Galois Hirata extensions. In this section, we shall characterize an $H^*$-Galois Hirata extension $B$ of $B^H$ in terms of the commutator subring $V_B(B^H)$ of $B^H$ in $B$ and the smash product $V_B(B^H)\# H$. A relationship between an $H^*$-Galois Hirata extension and an $H^*$-Galois Azumaya extension is also given. We begin with some properties of an $H^*$-Galois Hirata extension $B$ of $B^H$.

**Lemma 3.1.** If $A_1$ and $A_2$ are $H^*$-Galois extensions such that $A^H_1 = A^H_2$ and $A_1 \subseteq A_2$, then $A_1 = A_2$.

**Proof.** By Theorem 5.1 in [3], there exist $\{x_i, y_i \in A_1 | i = 1, 2, ..., n\}$ for some integer $n$ such that, for all $h \in H$, $\sum x_i(hy_i) = T(h)1_{A_1}$, where $T \in \mathbb{F}_H$, the set of right integrals in $H^*$. Let $t \in \mathbb{F}_H$, the set of left integrals in $H$, such that $T(t) = 1$, then $\{x_i, f_i = t(y_i) | i = 1, 2, ..., n\}$ is a dual basis of the finitely generated and projective right module over $A^H_1$. Since $A_1 \subseteq A_2$ such that $A^H_1 = A^H_2$, $\{x_i, f_i | i = 1, 2, ..., n\}$ is also a dual basis of the finitely generated and projective right module $A_2$ over $A^H_1$. This implies that $A_1 = A_2$.

**Lemma 3.2.** If $B$ is an $H^*$-Galois Hirata extension of $B^H$, then $B^H$ is a direct summand of $B$ as a $B^H$-bimodule.

**Proof.** We use the argument as given in [2]. Since $B$ is an $H^*$-Galois and a Hirata separable extension of $B^H$, $V_B(B^H)$ is a left $H$-Galois extension of $C$ ([8], Lemma 2.1 and Theorem 2.6). Hence $V_B(B^H)$ is a finitely generated and projective module over $C$ ([3], Theorem 2.2). Let $\Omega = \text{Hom}_C(V_B(B^H), V_B(B^H))$. Since $C$ is commutative, $V_B(B^H)$ is a progenerator of $C$. Thus $B$ is a left $\Omega$-module such that $B \cong V_B(B^H) \otimes C \text{Hom}_C(V_B(B^H), B) \cong V_B(B^H) \otimes C V_B(V_B(B^H))$ as $C$-algebras where $f(1) \in B^H$ for each $f \in \text{Hom}_C(V_B(B^H), B)$ by the proof of Lemma 2.8 in [2]. But $V_B(V_B(B^H)) = B^H$ ([2], Lemma 2.5), so $B \cong V_B(B^H) \otimes C B^H$. This implies that $V_B(B^H)$ is an $H^*$-Galois extension of $C$ ([2], Lemma 2.8); and so $C$ is a direct summand of $V_B(B^H)$ as a $C$-bimodule.
(2], Corollary 1.9 and Corollary 1.10). Therefore \(B^H\) is a direct summand of \(B\) as a \(B^H\)-bimodule.

By the proof of Lemma 3.2, \(V_B(B^H)\) is an \(H^*\)-Galois extension of \(C\).

**COROLLARY 3.3.** If \(B\) is an \(H^*\)-Galois Hirata extension of \(B^H\), then \(V_B(B^H)\) is an \(H^*\)-Galois extension of \(C\).

**COROLLARY 3.4.** If \(B\) is an \(H^*\)-Galois Hirata extension of \(B^H\), then \(B = B^H \cdot V_B(B^H)\) and the centers of \(B\), \(B^H\), and \(V_B(B^H)\) are the same \(C\).

**PROOF.** By Corollary 3.3, \(V_B(B^H)\) is an \(H^*\)-Galois extension of \(C\), so \(B^H \cdot V_B(B^H)\) is also an \(H^*\)-Galois extension of \(B^H\) (= \((B^H \cdot V_B(B^H))^H\)) with the same Galois system as \(V_B(B^H)\) ([3], Theorem 5.1). Noting that \(B^H \cdot V_B(B^H) \subset B\), we conclude that \(B = B^H \cdot V_B(B^H)\) by Lemma 3.1. Moreover, \(V_B(V_B(B^H)) = B^H\) ([8], Lemma 2.5), so the centers of \(B^H\), \(V_B(B^H)\), and \(B\) are the same \(C\).

**THEOREM 3.5.** Let \(B\) be an \(H^*\)-Galois extension of \(B^H\). The following statements are equivalent:

(1) \(B\) is a Hirata separable extension of \(B^H\),

(2) \(V_B(B^H)\) is an \(H^*\)-Galois extension of \(C\) and a direct summand of \(V_B(B^H) \# H\) as a \(V_B(B^H)\)-bimodule,

(3) \(V_B(B^H)\) is an Azumaya \(C\)-algebra and \(V_B(V_B(B^H)) = B^H\), and

(4) \(V_B(B^H)\) is an \(H^*\)-Galois extension of \(C\) and \(V_B(B^H) \# H\) is a direct sum of a finite direct sum of \(V_B(B^H)\) as a bimodule over \(V_B(B^H)\).

**PROOF.** (1) \(\iff\) (3) Since \(B\) is an \(H^*\)-Galois and a Hirata separable extension of \(B^H\), by Lemma 3.2, \(B^H\) is a direct summand of \(B\) as a \(B^H\)-bimodule. Thus \(V_B(V_B(B^H)) = B^H\) and \(V_B(B^H)\) is a separable \(C\)-algebra ([4], Proposition 1.3 and Proposition 1.4). But the center of \(V_B(B^H)\) is \(C\) by Corollary 3.4, so \(V_B(B^H)\) is an Azumaya \(C\)-algebra.

(3) \(\iff\) (1) Since \(V_B(B^H)\) is an Azumaya \(C\)-algebra and \(B\) is a bimodule over \(V_B(B^H)\), \(B \cong V_B(B^H) \otimes C V_B(V_B(B^H)) = V_B(B^H) \otimes C B^H\) as a bimodule over \(V_B(B^H)\) ([1], Corollary
3.6 on page 56). Noting that $B \equiv V_B(B^H) \otimes_C B^H$ is also an isomorphism as $C$-algebras and that $V_B(B^H)$ is an Azumaya $C$-algebra, we conclude that $V_B(B^H) \otimes_C B^H$ is a Hirata separable extension of $B^H$; and so $B$ is a Hirata separable extension of $B^H$.

(3) $\implies$ (2) By the proof of (3) $\implies$ (1) $B \equiv V_B(B^H) \otimes_C B^H$ such that $V_B(B^H)$ is a finitely generated and projective module over $C$, so $V_B(B^H)$ is an $H^*$-Galois extension of $C$ ([2], Lemma 2.8). Moreover, since $V_B(B^H)$ is an Azumaya $C$-algebra, $V_B(B^H)$ is a direct summand of $V_B(B^H) \otimes_C (V_B(B^H))^\circ$ as a $V_B(B^H)$-bimodule where $(V_B(B^H))^\circ$ is the opposite algebra of $V_B(B^H)$. But $V_B(B^H) \otimes_C (V_B(B^H))^\circ \equiv \text{Hom}_C(V_B(B^H), V_B(B^H)) \equiv V_B(B^H) \# H$ ([3], Theorem 2.2), so $V_B(B^H)$ is a direct summand of $V_B(B^H) \# H$ as a $V_B(B^H)$-bimodule.

(2) $\implies$ (3) Since $V_B(B^H)$ is an $H^*$-Galois extension of $C$, $B^H \cdot V_B(B^H)$ is an $H^*$-Galois extension of $(B^H \cdot V_B(B^H))^H$. But $(B^H \cdot V_B(B^H))^H = B^H$, so $B^H \cdot V_B(B^H)$ and $B$ are $H^*$-Galois extensions of $B^H$ such that $B^H \cdot V_B(B^H) \subset B$. Hence $B^H \cdot V_B(B^H) = B$ by Lemma 3.1. Thus the centers of $B$ and $V_B(B^H)$ are the same $C$. Moreover, $V_B(B^H)$ is a direct summand of $V_B(B^H) \# H$ as a $V_B(B^H)$-bimodule by hypothesis, so it is separable $C$-algebra ([3], Theorem 2.3). Thus $V_B(B^H)$ is an Azumaya $C$-algebra. But then $B \equiv V_B(B^H) \otimes_C V_B(B^H)$. On the other hand, by hypothesis, $V_B(B^H)$ is an $H^*$-Galois extension of $C$, so $B \equiv V_B(B^H) \otimes_C B^H$ ([2], Lemma 2.8). Therefore $V_B(V_B(B^H)) = B^H$.

(3) $\iff$ (4) Since $V_B(B^H)$ is an $H^*$-Galois extension of $C$, it is a finitely generated and projective module over $C$ and Hom$_C(V_B(B^H), V_B(B^H)) \equiv V_B(B^H) \# H$ ([3], Theorem 2.2). But then $V_B(B^H)$ is a Hirata separable extensions of $C$ if and only if $V_B(B^H) \# H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$ ([6], Corollary 3). Thus $V_B(B^H)$ is an Azumaya $C$-algebra if and only if $V_B(B^H)$ is an $H^*$-Galois extension of $C$ and $V_B(B^H) \# H$ is a direct summand of a finite direct sum of $V_B(B^H)$ as a bimodule over $V_B(B^H)$.

By Theorem 3.5, we can obtain a relationship between the class of $H^*$-Galois Hirata extensions and the class of $H^*$-Galois Azumaya extensions which were studied in [3] and
COROLLARY 3.6. Let $B$ be an $H^*$-Galois Azumaya extension of $B^H$. Then $B$ is an $H^*$-Galois Hirata extension of $B^H$ if and only if $C = C^H$.

**Proof.** ($\Rightarrow$) Since $B$ is an $H^*$-Galois Hirata extension of $B^H$, $V_B(B^H)$ is an Azumaya algebra over $C$ and a left $H$-Galois extension of $C$ ([8], Theorem 2.6). Hence $V_B(V_B(B^H)) = B^H$ ([8], Lemma 2.5). Thus $C \subset B^H$; and so $C = C^H$.

($\Leftarrow$) Since $B$ is an $H^*$-Galois Azumaya extension of $B^H$, $V_B(B^H)$ is separable over $C^H$ ([3], Lemma 4.1). Since $B$ is an $H^*$-Galois Azumaya extension of $B^H$ again, $V_B(B^H)$ is an $H^*$-Galois extension of $(V_B(B^H))^H$ ([3], Lemma 4.1), so both $B^H \cdot V_B(B^H)$ and $B$ are $H^*$-Galois extensions of $B^H$ such that $B^H \cdot V_B(B^H) \subset B$. Hence $B^H \cdot V_B(B^H) = B$ by Lemma 3.1. This implies that the center of $V_B(B^H)$ is $C$. But by hypothesis, $C = C^H$, so $V_B(B^H)$ is a separable extension of $C$. But $B = B^H \cdot V_B(B^H) \cong B^H \otimes_C V_B(B^H)$ as Azumaya $C$-algebras, so $B$ is a Hirata separable extension of $B^H$. Thus $B$ is an $H^*$-Galois Hirata extension of $B^H$.

COROLLARY 3.7. Let $B$ be an $H^*$-Galois Hirata extension of $B^H$. Then $B$ is an $H^*$-Galois Azumaya extension of $B^H$ if and only if $B$ is an Azumaya $C^H$-algebra.

**Proof.** ($\Rightarrow$) Since $B$ is an $H^*$-Galois Azumaya extension of $B^H$, $B^H$ is Azumaya $C^H$-algebra and $B$ is separable over $B^H$ ([3], Theorem 3.4). Hence $B$ is separable over $C^H$ by the transitivity of separable extensions. But $B$ is an $H^*$-Galois Azumaya extension of $B^H$ and an $H^*$-Galois Hirata extension of $B^H$ by hypothesis, so $C = C^H$ by Corollary 3.6. This implies that $B$ is an Azumaya $C^H$-algebra.

($\Leftarrow$) By hypothesis, $B$ is an Azumaya $C^H$-algebra. Hence $C = C^H$. But $B$ is an $H^*$-Galois Hirata extension of $B^H$, so $V_B(B^H)$ is an Azumaya subalgebra of $B$ over $C$ by Theorem 3.5-(3). Since $B$ is an $H^*$-Galois Hirata extension of $B^H$ again, $B$ is a Hirata separable extension of $B^H$ and a finitely generated and projective module over $B^H$. Thus $V_B(V_B(B^H)) = B^H$ ([5], Theorem 1); and so $B^H (= V_B(V_B(B^H)))$ is an Azumaya
subalgebra of $B$ over $CH$ by the commutator theorem for Azumaya algebras ([1], Theorem 4.3 on page 57). This proves that $B$ is an $H^*$-Galois Azumaya extension of $BH$.

4. Invariant subalgebras. For an $H^*$-Galois Hirata extension $B$, let $A$ be a subalgebra of $BH$ over $C$ such that $BH$ is a projective Hirata separable extension of $A$ and contains $A$ as a direct summand as an $A$-bimodule. In this section, we shall show that $V_{BH}(A)$ is the $H$-invariant subalgebra of a separable subalgebra $D$ in $B$ over $C$, that is, $D^H = V_{BH}(A)$. We denote by $S$ the set $\{A|A$ is a subalgebra of $BH$ over $C$ such that $BH$ is a projective Hirata separable extension of $A$ and contains $A$ as a direct summand as an $A$-bimodule}.

**Lemma 4.1.** Let $B$ be an $H^*$-Galois Hirata extension of $BH$. For any $A \in S$, $V_B(A)$ is an $H$-submodule algebra of $B$ and separable over $C$, and $(V_B(A))^H = V_{BH}(A)$ which is a separable $C$-algebra.

**Proof.** Since $A \in S$, $BH$ is a projective Hirata separable extension of $A$ and contains $A$ as a direct summand as an $A$-bimodule. But $B$ is an $H^*$-Galois Hirata extension of $BH$, so $B$ is a projective Hirata separable extension of $BH$. Hence, by the transitivity property of projective Hirata separable extensions, $B$ is a projective Hirata separable extension of $A$. Also $BH$ is a direct summand of $B$ as a $BH$-bimodule by Lemma 3.2, so $A$ is a direct summand of $B$ as an $A$-bimodule. Thus $V_B(A)$ is a separable algebra over $C$ ([5], Theorem 1). Moreover, it is clear that $(V_B(A))^H = V_{BH}(A) = V_{BH}(A)$, so $V_{BH}(A)$ is a separable $C$-algebra (Corollary 3.4 and [5], Theorem 1).

Next we want to show which separable subalgebra of $BH$ over $C$ is an $H$-invariant subring of an $H$-submodule algebra in $B$. Let $T = \{E \subset B|E$ is a separable $C$-subalgebra of $BH$ and satisfies the double centralizer property in $BH$ such that $V_{BH}(E) \in S\}$. Next we show that for any $E \in T$, $E$ is the $H$-invariant subring of an $H$-submodule algebra $D$ in $B$ which is separable over $C$.
THEOREM 4.2. Let $E$ be in $\mathcal{T}$. Then there exists an $H$-submodule algebra $D$ in $B$ which is separable over $C$ such that $D^H = E$.

PROOF. Since $E$ is in $\mathcal{T}$, $V_B^\pi(E)$ is in $S$ such that $V_B^\pi(V_B^\pi(E)) = E$. Now by Lemma 4.1, $V_B(V_B^\pi(E))$ is an $H$-submodule algebra of $B$ and separable over $C$ such that $(V_B(V_B^\pi(E)))^H = V_B^\pi(V_B^\pi(E))$. But $V_B^\pi(V_B^\pi(E)) = E$, so $(V_B(V_B^\pi(E)))^H = E$. Let $D = V_B(V_B^\pi(E))$. Then $D$ satisfies the theorem.

By Theorem 4.2, we obtain an expression for the separable $H$-submodule algebra $D$ for a given $E$ in $\mathcal{T}$.

COROLLARY 4.3. By keeping the notations as given in Theorem 4.2, let $Z$ be the center of $E$. Then $D \cong E \otimes_Z V_D(E)$ as Azumaya $Z$-algebras.

PROOF. Since $E$ satisfies the double centralizer property in $B^H$, $V_B^\pi(V_B^\pi(E)) = E$. Hence the centers of $E$ and $V_B^\pi(E)$ are the same $Z$. Similarly as given in the proof of Lemma 4.1, since $V_B^\pi(E)$ is in $S$, $B (= B^H : V_B(B^H))$ is a projective Hirata separable extension of $V_B^\pi(E)$ and contains $V_B^\pi(E)$ as a direct summand as a $V_B^\pi(E)$-bimodule by the transitivity property of projective Hirata separable extensions and the direct summand conditions. Thus $V_B^\pi(E)$ satisfies the double centralizer property in $B$, that is, $V_B(V_B^\pi(E)) = V_B^\pi(E)$. This implies that the centers of $V_B^\pi(E)$ and $V_B(V_B^\pi(E))$ are the same. Therefore $D$ and $E$ have the same center $Z$. Noting that $D$ and $E$ are separable $C$ algebras by Theorem 4.2, we conclude that $E (= D^H)$ is an Azumaya subalgebra of $D$ over $Z$, and so $D \cong E \otimes_Z V_D(E)$ as Azumaya $Z$-algebras ([1], Theorem 4.3 on page 57).

Remark. When $B$ is an $H^*$-Galois Azumaya extension of $B^H$, the correspondence $A \rightarrow V_B(A)$ as given in Lemma 4.1 recovers the one-to-one correspondence between the set of separable subalgebras of $B^H$ and the set of $H^*$-Galois extensions in $B$ containing $V_B(B^H)$ as given in [3].
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