Proposal for Caterpillar Fellowship (2008-2009)

Generalized Differentiation on Metric Spaces

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1. Introduction

Modern variational analysis can be viewed as an outgrowth of two major branches of applied mathematics: calculus of variations and mathematical programming. The main application of variational analysis is to solve optimization problems. Existing variational analysis is developed primarily in Banach spaces. The reason is that the basic concepts (directional derivative and gradient) in analysis mandate the notions of direction and dual space of Banach spaces. However, there are important optimization problems that are naturally formulated on metric spaces and whose solution requires a variational analysis on metric spaces.

To develop a variational analysis on metric spaces, one has to overcome the fundamental obstacle that general metric spaces do not have the notions of direction and dual space which are needed to define directional derivative and gradient. Thanks to previous support from Caterpillar Fellowship at Bradley, we have succeeded in generalizing some concepts of differentiation to metric spaces in [4] [6] [7] and [9]; see p.3 for a brief description. Here we propose a systematic study on generalized differentiation on metric spaces, which is an essential step towards the ultimate goal of developing a variational analysis on metric spaces.

The outcomes of this proposed research shall be mathematically significant and practically applicable. The Caterpillar Fellowship, if awarded, would enable me to concentrate on this project during the next summer break, working with my collaborators, preparing the results for publication, and exploring/applying for external funding. It would also provide the fund for inviting outside researchers to Bradley to give lectures and exchange ideas related to the proposal.
2. Problem Statement and Related Recent Work

A. Problem Statement.

The main motivation of this research is the following general optimization problem on metric spaces. Let \( J \) be a function on a complete metric space \((X, d)\), \(S(\cdot)\) a map from \(X\) to a Banach space \((Y, \| \cdot \|)\), and \(Q \subset Z\) a given closed subset.

Optimization Problem

Optimize \( J(w), w \in X \) subject to \( S(w) \in Q \). \( \quad (1) \)

If \( X \) is a Banach space, then the solutions of problem (1) can be described in terms of the directional derivatives and gradients of \( J \) and \( S \). In this case the vector structure and dual space \( X^* \) (the space of linear functionals on \( X \)) of \( X \) are dispensable to the definitions of directional derivative and gradient. Indeed, if a function \( f : (X, \| \cdot \|) \to \mathbb{R} = (-\infty, \infty) \) is (Frechet) differentiable at \( x \), then the gradient \( \nabla f \in X^* \) is defined by

\[
\langle \nabla f(x), v \rangle = f'(x; v)
\]

for all \( v \in X \), where \( f'(x; v) \) is the directional derivative of \( f \) at \( x \) in the direction of \( v \), that is

\[
f'(x; v) = \lim_{t \to 0} \frac{f(x + tv) - f(x)}{t}.
\]

If \( f \) is not necessarily differentiable but Lipschitz, then a generalized directional derivative \( f'(x; v) \) can still be defined as in [2] and a generalized gradient \( \xi \in X^* \) of \( f \) at \( x \) satisfies

\[
\langle \xi, v \rangle \leq f'(x; v)
\]

for all \( v \in X \). The set of all generalized gradients of \( f \) at \( x \) is called the subdifferential of \( f \) at \( x \). The classical Hahn-Banach theorem guarantees that the subdifferential is nonempty.

So it is evident that the linear structure of \( X \), the dual space \( X^* \), and Hahn-Banach theorem play essential roles in the definitions of (generalized) gradient and subdifferential.

A general metric space does not have linear structure nor a notion of dual space. How can directional derivative and gradient be defined? How can they be related to solutions of optimization problems? These are two of the many issues we need to address in order to develop a variational analysis on metric spaces.

B. Recent Related Work.

Progress has been made in my recent works [4] [6] [7] and [9] (some with others) in developing a variational analysis on metric spaces. We review some related results as follows.
1. Derivates and multiplier rules. In [4] a generalized directional derivative, called *derivate*, and a strict derivate of $J$ and $S$ are defined. In terms of the strict derivate $D_s(J, S)\{w_0\}$ of the joint map $(J, S) : X \to \mathbb{R} \times Y$, the following multiplier rule is proved for solutions of Optimization Problem (1) above.

**Theorem 1. (Multiplier Rule).** Suppose that $w_0$ is a minimum point of $J(\cdot)$ subject to $S(\cdot) \in Q$. Suppose that $Y$ has strictly convex dual $Y^*$ and $Q \subset Y$ is closed, convex and finitely codimensional. Then there exists nontrivial $(\psi^0, \psi) \in [0, \infty) \times Y^*$ such that

\[
\begin{align*}
\psi^0 y^0 + \langle \psi, y \rangle_{Y^*, Y} &\geq 0 \text{ for all } (y^0, y) \in D_s(J, S)\{w_0\}, \quad (2.1) \\
\langle \psi, \eta - S(w_0) \rangle_{Y^*, Y} &\leq 0 \text{ for all } \eta \in Q. \quad (2.2)
\end{align*}
\]

In [7] this multiplier rule is generalized in terms of a generic subdifferential of distance function of $Q$, which can be non-convex.

2. An application of the multiplier rule on metric spaces

In [5], we apply the multiplier rule stated above to give a unified and direct proof of the maximum principle for optimal controls with isoperimetric and pointwise constraints under the most general differentiability assumption on the data.

3. Hahn-Banach extension theorem on cones

As mentioned above, Hahn-Banach extension theorem is essential to the proof of the existence of gradients. In [6], an extension theorem of Hahn-Banach type is proved for linear functions on convex cones of a Banach space. This result is used then to prove the existence of the subdifferential of a Lipschitz function defined on a subset of a Banach space. This extension theorem is very significant. It shows that the dual spaces of the tangent cones of a metric space are the appropriate substitutes for the dual space of a Banach space needed to define subdifferentials.

4. Subdifferential of convex functions on metric length space

The paper [9] a subdifferential is defined for convex functions on a metric length space with tangent spaces isometric to convex cones. As an application, the classical Helly's theorem on Euclidean spaces (proved in 1923) and on manifolds of nonpositive curvature (proved in 2005) is generalized to metric spaces in [9].
3. Proposed Projects and Methodologies

Here we propose a systematic study on generalized differentiation on metric spaces. The focus will be on the following subjects.

- Systematic study on derivates on metric spaces
- Relative subdifferentials of functions on subsets of Banach spaces
- Subdifferentials of functions on metric length spaces
- Applications of generalized differentiation

A. Systematic study on derivates on metric spaces

Derivates are introduced in [4] and [7] as generalized directional derivative on metric spaces. The multiplier rules proved in these papers show that the notions of derivates are appropriate and useful. In order to fully utilize these concepts in variational analysis, a systematic study is needed. In this proposal, we will address the following issues.

Calculus rules of derivates. We will examine whether derivates obey the common calculus rules such as sum rule, chain rule and so on.

Other versions of derivates. We will also consider \( \delta \)-derivates, sided derivates, their properties and applications.

Relationship with subdifferentials. This relationship enables us to reformulate multiplier rules in terms of subdifferentials instead of derivates.

B. Relative subdifferentials of functions on subsets of Banach spaces

In [6] a notion of relative subdifferential in Clarke's sense is defined for Lipschitz functions on a subset of a Banach space. In this proposal we will consider relative subdifferentials in other senses. Consider a subset \( W \) of a Banach space \( (X, \| \cdot \|_X) \). For possible broader applications, we assume that \( W \) be equipped with another metric \( d \) satisfying \( \| x - y \|_X \leq d(x, y) \) for all \( x, y \in W \); this means that the inclusion \( i : (W, d) \rightarrow (X, \| \cdot \|_X) \) is continuous. Let \( F : W \rightarrow \mathbb{R} \cup \{ \infty \} \) be a function and \( \varepsilon > 0 \). We propose a definition of an \( \varepsilon \)-subdifferential \( \partial F(x; W) \) of \( F \) at \( x \in W \) as follows

\[
\tilde{\partial}_\varepsilon F(x; (W, d)) = \left\{ x^* \in X^* \mid \liminf_{u \to x} \frac{F(u) - F(x) - \langle x^*, u - x \rangle}{d(u, x)} \geq -\varepsilon \right\},
\]

where \( \langle \cdot , \cdot \rangle \) is the pairing between \( X^* \) and \( X \), and \( u \xrightarrow{W} x \) means that \( u \in W \) and \( d(u, x) \to 0 \). The limit \( \partial F(x; W) = \lim_{x \to x, \varepsilon \downarrow 0} \tilde{\partial}_\varepsilon F(x; (W, d)) \) is called the basic/limiting subdifferential of \( F \) at \( x \).
Remark 1. If $F$ is constant, then (3) reduces to
\[
\partial_{\varepsilon} F(x; (W, d)) = \left\{ x^* \in X^* \mid \limsup_{W \to x} \frac{\langle x^*, u - x \rangle}{d(u, x)} \leq \varepsilon \right\},
\]
which generalizes the notion of $\varepsilon$-normals of $W$ defined in [8].

Remark 2. Definition (3) easily extends to a set-valued map $F$ from $(W, d)$ to another Banach space $(Y, \| \cdot \|_Y)$. The $\varepsilon$-subdifferential of $F$ at $(x, y) \in \text{gph}(F)$ is the set-valued map $\partial_{\varepsilon} F((x, y); (W, d)) : Y^* \to X^*$ such that
\[
\partial_{\varepsilon} F((x, y); (W, d))(y^*) = \left\{ x^* \in X^* \mid \lim_{(u, v) \in \text{gph}(F) \to (x, y)} \inf \frac{\langle y^*, v - y \rangle - \langle x^*, u - x \rangle}{d(u, x)} \geq -\varepsilon \right\},
\]
for each $y^* \in Y^*$, where $\langle x^*, u - x \rangle$ and $\langle y^*, v - y \rangle$ are pairings between $X^*$ and $X$, $Y^*$ and $Y$, respectively. If $F$ is single-valued, then $(u, v) \in \text{gph}(F)$ means that $v = F(u)$.

Remark 3. By adding $\|F(u) - F(x)\|_W, \|v - y\|_W$ and $\|F(u) - F(x)\|_W$ to $d(u, x)$ in (3)-(0), we obtain the definitions of the corresponding geometric $\varepsilon$-subdifferentials of $F$.

Remarks 1-3 show that various notions of normals and subdifferentials can be generalized and unified under subdifferentials relative to a subset of the Banach space. We will study the relationships among such subdifferentials, the derivates of maps on $W$, and the geometry (tangents/normals) of $W$. These relationships will translate the multiplier rules in terms of subdifferentials and lead to general necessary conditions for optimization problems like (1).

C. Subdifferentials of functions on metric length spaces

On a general metric space $(M, d)$, the notions of curve (or path), length (of a curve), length metric, angle (between curves), direction, tangent spaces, convexity and other geometric terms can all be introduced, though they may not have the expected properties of such terms in common senses; see [1]. On a metric length space, such as a Riemannian manifold, $d(x, y)$ is the same as the induced length metric (the infimum of the lengths of all curves connecting $x$ and $y$). In [9] the notion of subdifferential is introduced for convex functions defined on a length metric space whose tangent spaces are isometric to cones. The Hahn-Banach extension theorem in [6] is used to show that the subdifferential does exist. A nice application of this subdifferential is a generalization of the classical Helly's theorem to general metric length spaces [9].

Lipschitz functions are the most useful functions in variational analysis. In this proposal, we are going to define subdifferentials for Lipschitz functions on metric spaces. This requires a thorough understanding on the space of the tangent spaces (i.e., tangent bundle). For example, an
analogue of the generalized directional derivative in Clarke's sense will involves "tangent vectors" at points nearby. So we need to know how are the tangent cones related to each other.

As shown in [9], a variational analysis on metric length spaces may have various applications to geometric problems.

D. Applications of generalized differentiation

Variational analysis on metric spaces differs from and complements to the "classical analysis" currently developed on metric spaces. In addition, variational analysis is strongly motivated by the need of solving optimization problems. Therefore, possible applications will be investigated as we focus on a systematic study on the generalized differentiation on metric spaces.

4. Expected Outcomes

This project will result in a series of papers, publishable in professional mathematical journals. The outcomes of this project may help secure external funding for developing a complete variational analysis on metric spaces. This research has been an excellent collaborative opportunity for me with my colleagues at and outside Bradley. Summer break is an ideal time for us to work together on this project. In addition, this research provides appropriate topics for our courses Mth494/495 (Senior Project in Mathematics) and Mth490/501/502 (Topics in Mathematics and Applied Mathematics).

5. References Cited