An Elementary Proof that the Borromean Rings are Non-Splittable

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Linström and Zetterström [1] gave a proof that the Borromean rings (figure 1) could not consist of true circles. In this note, we give an elementary proof (sans algebraic topology) that the Borromean rings are "linked" though no two components are. The tool that we use is the colorability *mod n* of a knot or link diagram. This tool has been presented in honors undergraduate seminars. I have included a discussion of colorability *mod n* though the technique is well known. For example, see Kauffman, Chapter VI [2].

1. DEFINITIONS. A *knot* will be defined as a smooth (or polyhedral) simple closed curve in 3-space R^3. A *link* is defined as a collection of disjoint smooth (or polyhedral) simple closed curves in R^3. Two knots or links K_1 and K_2 are said to be *equivalent* if there is an orientation preserving homeomorphism h: $R^3 \to R^3$ such that $h(K_1) = K_2$. A link L is said to be *splittable* if there exists a smooth (or polyhedral) 3-ball B, an ordering of the components of the link $K_1, K_2, \dots, K_m$ and an integer $0 < k < m$ such that $K_j \subset B$ for $j \leq k$ and $K_i \subset S^3 - B$ for $i > k$. A *diagram* for a knot or link K is an image of a regular projection (all self-intersections are non-tangential (transverse) and are double points) of K onto a plane with crossing information at each double point (p. 215, reference 2). Note that FIGURES 1 and 4 are examples of diagrams. Two knot or link diagrams D_1 and D_2 are said to be *equivalent* if D_1 can be obtained from D_2 by:

1) Deformations of the plane which do not alter the crossing information at each double point and

2) The three Reidemeister moves and their inverses. See FIGURE 2 for an illustration of these.

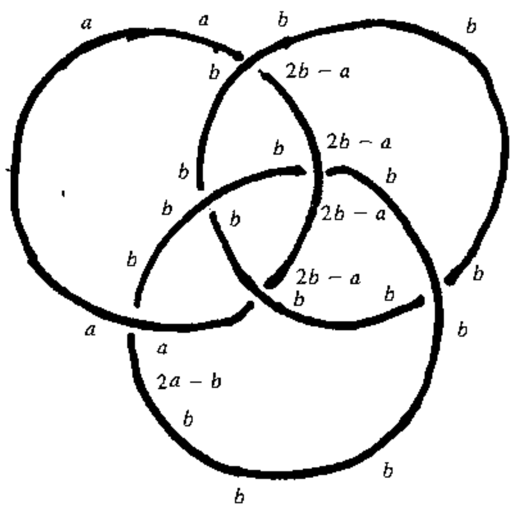

Figure 1

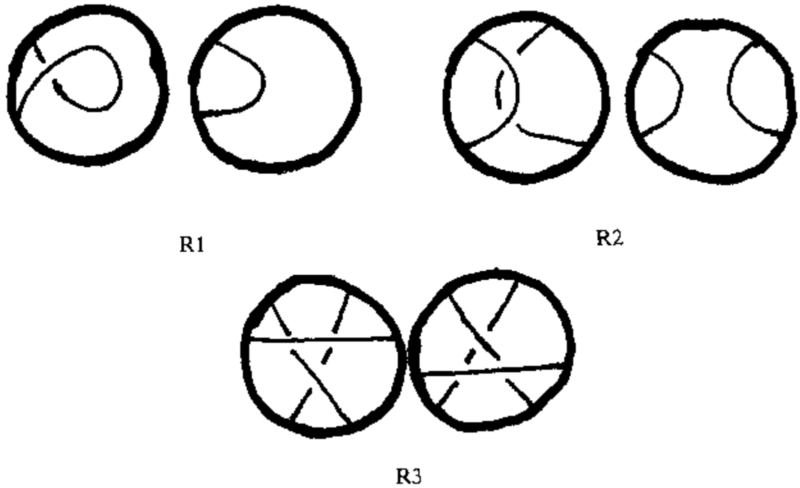

Figure 2. The Reidemeister Moves

2. THEOREMS. The following theorem is well known and will not be proved here.

Theorem 1. *Two knots or links are equivalent if and only if they have equivalent diagrams. See section* $1B$ *of reference* [**4**] *for a proof.*

A knot or a link K is said to be *colorable* mod n (n is assumed to be 3 or greater) if K has a diagram D in which it is possible to assign an integer to each arc of D which does not contain an undercrossing of D such that:

1) at each crossing we have $a + c = 2b \pmod n$ where b is the integer assigned to the overcrossing and a and c are the integers assigned to the other two arcs (see Figure 3) and

2) at least 2 distinct integers mod n are used in the diagram.

The following theorem is well known:

Theorem 2. *If* K_1 *is a knot or a link which is colorable* mod n *then every diagram of* K_1 *is colorable* mod n.

Proof: Exercise. All one has to check is: if a diagram D is colorable mod n and if one applies either a Reidemeister move (or its inverse) to D, the resulting diagram remains colorable mod n. □

It follows from Theorem 1 and Theorem 2 that if K_1 is a knot or a link which is colorable mod n and K_2 is equivalent to K_1, then K_2 is colorable mod n.

Corollary 3. *There exists a knot which is not equivalent to the unknot.*

Proof: Note that the trefoil knot (see Figure 4) is colorable mod 3 whereas the unknot is not. □

We now come to the main result of this note:

Theorem 5. *If a link L is splittable then L is colorable* mod 3.

Proof: If L is splittable with a splitting ball B, then there exists a diagram for L in which the images of $L \cap B$ are separated from the images of $L \cap (S^3 - B)$ by a circle C. Give the components of the diagram of $L \cap B$ the monochrome coloring by assigning the integer 0 to each strand. Similarly, assign the strands of the diagram of $L \cap (S^3 - B)$ the integer 1. □

It is an exercise to see that the standard diagram of the Borromean rings is not colorable mod n for any $n > 1$. The integer labeling of the diagram depicted in FIGURE 1 illustrates this: one has no choice but to set $a = b$. Thus we have an elementary proof that the Borromean rings link is unsplittable and thus the rings cannot be pulled apart.

Remark. If a knot or link K is colorable mod n, then one can obtain a homomorphism from $\pi_1(R^3 - K)$ onto the dihedral group D_n where $D_n = \{s, t | s^2 = 1 = t^n, sts = t^{n-1}\}$. This homomorphism is determined by the particular choice of coloring. See Kaufman [2] or Fox [5].

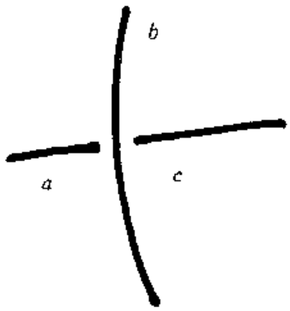

Figure 3

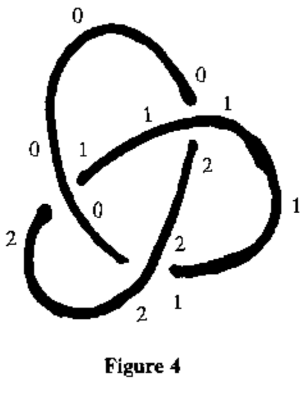

Figure 4

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Letter to the Editor:

Recently Grosof and Taiani [1] gave an algebraic proof that if $Q(X) = \prod_1^n (X - r_i)$ with the r_i distinct, then $\Sigma P(r_i)/Q'(r_i) = 0$ for $\deg(P) \leq n - 2$. I should like to add that this result has a home in algebraic number theory, as part of the computation of the "different". The usual proof there [2, p. 135; 3, p.56; 4, p.144] is yet another ingenious algebraic argument. First, standard methods yield the partial fraction decomposition

$$1/Q(X) = \Sigma Q'(r_i)^{-1}/(X - r_i).$$

The right-hand side, as a formal power series in X^{-1}, is

$$X^{-1}\Sigma Q'(r_i)^{-1}/(1 - r_i X^{-1}) \\ = \Sigma[Q'(r_i)^{-1} r_i^k](X^{-1})^{k+1}.$$

But the left-hand side is

$$(X^n + a_1 X^{n-1} + \ldots)^{-1} \\ = X^{-n}(1 + a_1 X^{-1} + \ldots)^{-1} \\ = X^{-n} - a_1 X^{-(n+1)} + \ldots$$

Comparing terms, we recover the fact that $\Sigma r_i^k/Q'(r_i) = 0$ for $k \leq n - 2$; we also see that the sum is equal to 1 when $k = n - 1$.

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