C^1 Paths Detect Continuity for Two-Variable Functions

A common way to show that a function of two variables is not continuous at a point is to show that the one dimensional limit of the function evaluated over a curve varies according the curve that is used. For example one can show that the function $f(x,y) = \begin{cases} \frac{xy}{(x^2+y^2)}, (x,y) \neq (0,0) \\ 0 \text{ if } (x,y) = (0,0) \end{cases}$ is discontinuous at (0,0) by showing that $\lim_{x \to 0} f(x, mx) = \frac{m}{1+m^2}$ which varies with m. One then gives the caveat that the natural converse to this technique cannot be used to demonstrate that a function is continuous. One reminds the students that $\lim_{(x,y)\to(a,b)} f(x,y)$ exists only when the limit of f exists as (x,y) approaches (a,b) over *all* curves that run through (a,b). There is often some vagueness as to what is meant by *all* curves (e. g., all continuous curves, all differentiable curves) and such vagueness can lead to trouble. For example, [1] (example 3, chapter 9, $f(x,y) = \begin{cases} \frac{(exp(-1/x^2)y}{exp(-2/x^2)+y^2}, (x,y) \neq (0,0) \\ 0, (x,y) = (0,0) \end{cases}$ gives an example of a function $f(x,y)$ where $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist and yet $\lim_{x\to 0} f(x, g(x))$ for all algebraic functions of the type $g(x) = mx^r$, r rational and positive. In [2] an example is given of a function $f(x,y)$ where $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist and yet $f(x,y)$ is continuous on all of $\mathbb{R}^2 - \{(0,0)\}$ and $\lim_{x\to 0} f(x, g(x)) = 0$ for all analytic functions g where $g(0) = 0$. In other words, analytic and algebraic paths are not sufficient to detect discontinuity.

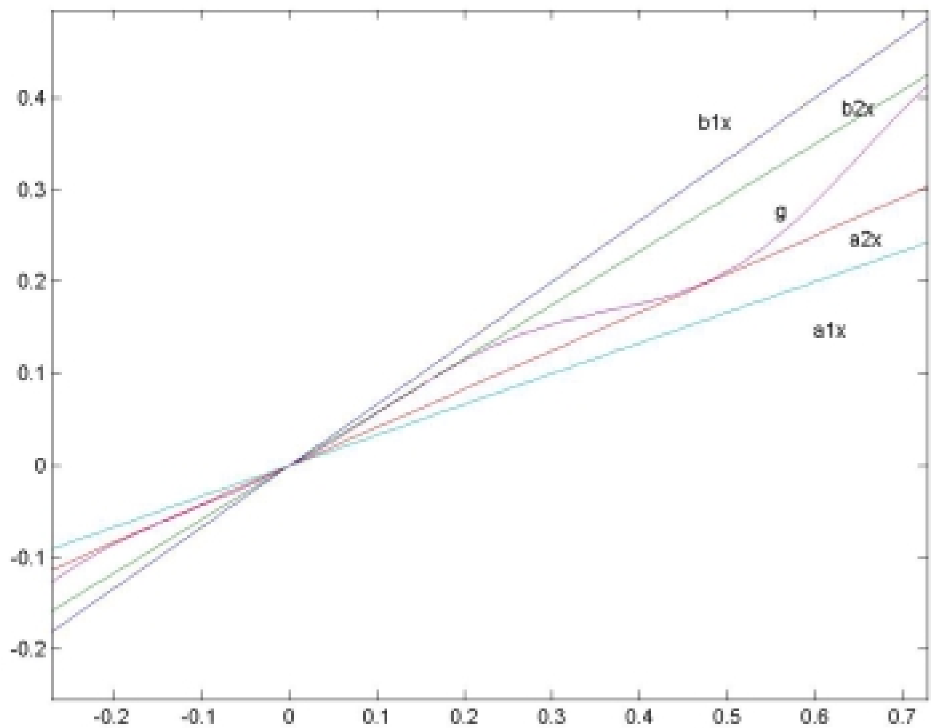
In this paper we show that C^1 paths (continuous first derivative) are sufficient to detect discontinuity; that is, if $\lim_{x\to 0} f(x, g(x)) = L = \lim_{y\to 0} f(h(y), y)$ for all C^1 functions g, h that run through the origin, then f is continuous there and $\lim_{(x,y)\to(0,0)} f(x,y) = L$ The author has tried (and failed, so far) to extend this result to C^n $(n > 1)$ or to find a counterexample for $n < \infty$.

We start by proving a technical lemma that will be used in the proof of the main theorem.

Differentiability Lemma.
Suppose
1)$g(0) = 0$
2) There exists a family of slopes $\{a_i\}, \{b_i\}$ and associated neighborhoods $[-\delta_i, \delta_i]$ where, for all $i \in \{1, 2, ...\}$,
a) $a_1 \leq a_2 \leq ...a_i \leq a_{i+1}....m... \leq b_{i+1} \leq b_i \leq ...b_2 \leq b_1$ and $a_i \to m$ and $b_i \to m$ and
b) $0 < ..\delta_{i+1} < \delta_i < .. < \delta_1$ where $a_i x \leq g(x) \leq b_i x$ for $0 < x < \delta_i$, and
$a_i x \geq g(x) \geq b_i x$ for $-\delta_i < x < 0$.

Then $g'(0)$ exists and $g'(0) = m$.

Proof. See Figure One. Let $\epsilon > 0$ be given. For now assume $x > 0$. Choose N such that for all $i > N, b_i - a_i < \frac{\epsilon}{2}$. Then there is a $\delta_i > 0$ such that for $0 < x < \delta_i$, $a_i x \leq g(x) \leq b_i x$.
Then
$a_i \leq \frac{g(x)}{x} \leq b_i \to a_i - m \leq \frac{g(x)}{x} - m \leq b_i - m \to -\epsilon \leq \frac{g(x)}{x} - m \leq \epsilon \to \lim_{x \to 0+} \frac{g(x)}{x} = m$
$\to \lim_{x \to 0+} \frac{g(0+x) - g(0)}{x} = m.$
The argument for $x < 0$ is similar.



Main Theorem.
Suppose for all C^1 functions g and h where $g(0) = 0 = h(0)$ we have $\lim_{x \to 0} f(x, g(x)) = L$ and $\lim_{y \to 0} f(h(y), y) = L$. Then $\lim_{(x,y) \to (0,0)} f(x, y) = L$.
Note: by [2], this result is false if "C^1" is replaced by "real analytic".

Proof. Assume that f is not continuous at $(0, 0)$. We can assume that for each m, $\lim_{x \to 0} f(x, mx) = L = \lim_{y \to 0} f(0, my)$. Otherwise we are done. We will construct a C^1 function g such that $g(0) = 0$ and that either $\lim_{x \to 0} f(x, g(x)) \neq L$ or $\lim_{y \to 0} f(h(y), y) \neq L$. Since $\lim_{(x,y) \to (0,0)} f(x, y)$ does not exist, we can assume there is some fixed $\epsilon > 0$ and some sequence of points (x_i, y_i) (denoted by p_i) where $p_i \to (0,0)$ as $i \to \infty$ and $|f(x_i, y_i) - L| > \epsilon$. Let S denote $\{p_i\}$.

Divide the plane into 8 wedge shaped regions bounded by the lines $x = y, x = -y, x = 0, y = 0$. None of these lines can contain more than a finite number of points of S so we can assume for this argument that none of them contain any. But at least one of the regions must contain an infinite number of points of S; assume it is the region in the first quadrant bounded above by the line $y = x$ and below by the line $y = 0$. Denote this region by W_1. So, we can modify S as necessary (by passing to a convergent sub sequence) so that all of its points lie in W_1. Let h_1 denote the function $h_1(x) = m_1 x$ where m_1 is the slope of the line running through $(0,0)$ and p_1. Set $q_1 = p_1.,$ and $a_1 = 0, b_1 = 1$. Note that $b_1 - a_1 = 1$. For "bookkeeping" purposes, let $t_1 = m_1$; the meaning of t_i will become clear later on.

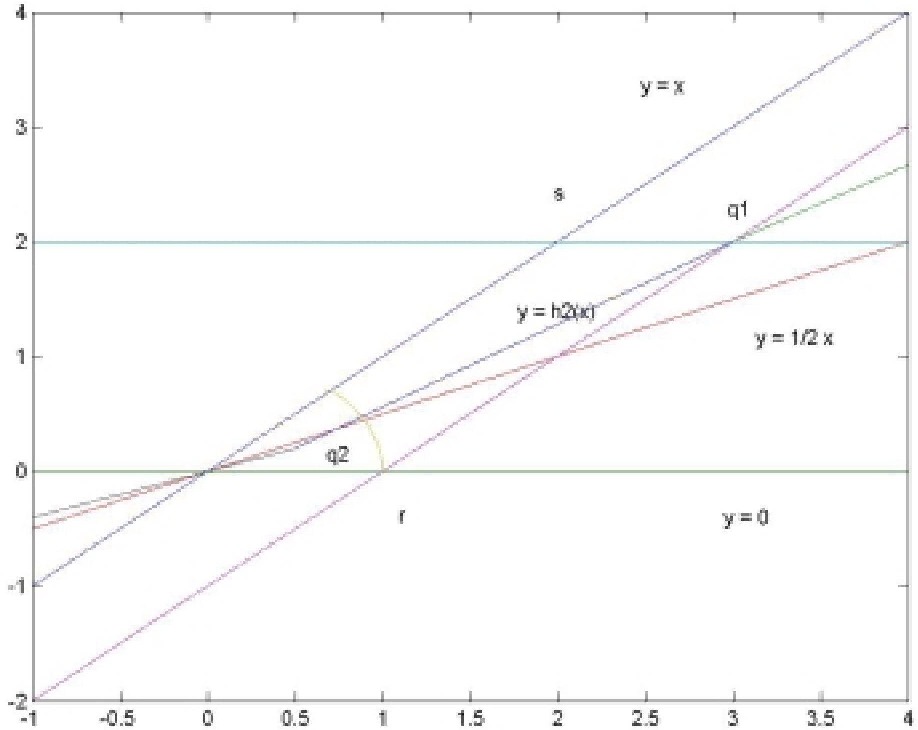
Now consider the line $y = \frac{1}{2}x$. Set $n_1 = \frac{1}{2}$. This line splits W_1 into two parts, at least one of which contains an infinite number of points of S. Call the chosen "half wedge" W_2 . Run a line though q_1 that is parallel to the line $y = x = b_1 x$. This line intersects the line $y = a_1 x = 0$ at some point r. Run a second line through q_1 which is parallel to the line $y = 0 = a_1 x$. This line intersects the line $y = b_1 x$ at some point s. Let B_1 denote the intersection of the ball of radius δ_1 centered at $(0,0)$ with W_1 where $\delta_1 = min(||r||, ||s||)$. $B_1 \cap W_2$ contains an infinite number of points of S; choose $q_2 = (x_2, y_2)$, $q_2 \in B_1 \cap W_2 \cap S$. If q_2 is below the line $y = n_1 x$ then set $[a_2, b_2] = [0, n_1 x] = [0, \frac{1}{2}]$; if q_2 is above set $[a_2, b_2] = [n_1, 1] = [\frac{1}{2}, 1]$.

Note that $b_2 - a_2 = \frac{1}{2}$. Let m_2 denote the slope of the line from $(0,0)$ to q_2. Now consider the function

$$h_2(x) = \begin{cases} m_2 x, x \leq x_2 \\ t_2 x + b, x_2 < x \leq x_1 \\ h_1(x), x > x_1 \end{cases} \quad \text{where } t_2 = \frac{(y_2 - y_1)}{(x_2 - x_1)} \text{ and } b = -t_2 x_1 + y_1$$

(that is, the middle part of the graph of h_2 is a line segment connecting q_2 to q_1.)

Notice that $h_2'(0) = m_2$ and that for all $x < x_2$, $\frac{h_2(x)}{x} = m_2 \in [a_2, b_2]$. Also, note that $a_1 \leq t_1 \leq b_1$. See Figure Two.



We now seek to define $h_k(x)$ for all $k \in \{1, 2, ...\}$ inductively.
Of course $a_i, b_i, q_i = (x_i, y_i)$, W_k, t_i and h_i have all been defined for $i \in \{1, 2, ..k\}$.
Run a lines l_u and l_l though q_k where l_u is parallel to the line $y = a_k x$ and the line l_l is parallel to the line $y = b_k x$. l_u intersects the line $y = b_k x$ at a point s and the line l_l intersects the line $y = a_k x$ at the point r. Let $\delta_k = min\{||r||, ||s||\}$.
Let B_k denote the ball centered at the origin of radius δ_k. $B_k \cap W_k$ contains an infinite number of points of S that do not intersect the line $y = m_{k+1}x$ where $m_{k+1} = \frac{1}{2}(b_k + a_k)$. Pick one can denote it by q_{k+1}. If $q_{k+1}$is above the line $y = m_{k+1}x$ then let $a_{k+1} = m_{k+1}$ and $b_{k+1} = b_k$. Else let $a_{k+1} = a_k$ and $b_{k+1} = m_{k+1}$.

Now define the function:

$$h_{k+1} = \begin{cases} m_{k+1}x, x \le x_{k+1} \\ t_{k+1}x + b,\ x_{k+1} < x \le x_k \text{ where } t_{k+1} = (\frac{y_{i+1} - y_i}{x_{i+1} - x_i}) \text{ and } b = -t_{k+1}x_k + y_k \\ h_k(x), x > x_k \end{cases}$$

Note that $a_{k+1} < m_{k+1} < b_{k+1}$. Note that $[a_{k+1}, b_{k+1}] \subset [a_k, \frac{b_k + a_k}{2}]$ or $[a_{k+1}, b_{k+1}] \subset [\frac{b_k + a_k}{2}, b_k]$ and $b_{k+1} - a_{k+1} = \frac{1}{2}(\frac{1}{2^{k-1}}) = \frac{1}{2^k}$.
Now h_k, a_k and b_k have been defined for all k. By the nested interval theorem, there exists some number m, $0 \le m \le 1$, $a_k \to m$ and $b_k \to m$ as $k \to \infty$.

Now define $h(x) = \begin{cases} mx, x \leq 0 \\ h_k(x), x_k \leq x < x_{k-1} \quad k \in \{2,3,...\} \\ t_1 x, x \geq x_1 \end{cases}$

That is, the graph of $h(x)$ for $x \in (0, x_1)$ consists of straight line segments of slope t_{i+1} running from points (x_{i+1}, y_{i+1}) to (x_i, y_i).
Note that h is continuous over $(-\infty, \infty)$. Note that by the Differentiability Lemma, $h'(0) = m$ because for $0 < x < x_k$, $a_k x < h(x) < b_k x$ and $h(x) = mx$ for $x < 0$.

Therefore, h is differentiable at $x = 0$. Of course, h might not be differentiable at each x_i, $i \in \{1, 2, ...\}$. So, we seek to modify h by "rounding" at each "corner" by a cubic spline. We choose the cubic spline method rather than a traditional "bump function" because we wish to control the first derivative as $x \to 0+$.

Spline calculation.

Lemma. The cubic $P(x) = (\frac{m_{i+1} - m_i}{\epsilon^2})x^3 - 2(\frac{m_{i+1} - m_i}{\epsilon})x^2 + m_{i+1}x$
has the following properties:
$P(0) = 0, P(\epsilon) = m_i\epsilon, P'(0) = m_{i+1}, P'(\epsilon) = m_i,$
$max_{x \in [0,\epsilon]}|P'(x)| = max\{|m_i|, |m_{i+1}|, \frac{1}{3}|(4m_i - m_{i+1})|\}$
Proof. Routine calculation.

Note: if it is the case that $m_i, m_{i+1} \in (0,1)$ and $|m_{i+1} - m_i| < \frac{1}{2^i}$
Then $\frac{1}{3}|4m_i - m_{i+1}| = \frac{1}{3}|3m_i + m_i - m_{i+1}| \leq \frac{1}{3}3|m_i| + \frac{1}{3}|m_i - m_{i+1}|$
$\leq m_i + \frac{1}{3}\frac{1}{2^i}$
But also $\frac{1}{3}|3m_i + m_i - m_{i+1}| > \frac{1}{3}|3m_i - \frac{1}{2^i}| = |m_i - \frac{1}{3(2)^i}| = m_i - \frac{1}{3} * \frac{1}{2^i}$

In short, the derivative cannot change sign and $max_{x \in [0,\epsilon]}|P'(x)| = max\{|m_i|, |m_{i+1}|\}$

Lemma. For $x \in [0, \epsilon]$, the graph of $P(x)$ cannot leave the disk $D_\epsilon((0,0)) = \{(x,y),$
$0 \leq x^2 + y^2 \leq \epsilon\}$.
Proof. Due to the fact that $P'(x)$ does not change sign, the graph of $P(x)$ stays inside the rectangle formed the lines $y = 0$ and $y = m_i\epsilon$, $x = 0$, $x = \epsilon$ which is a subset of $D_\epsilon((0,0))$.

Corollary. The cubic
$P_{i+1}(x) = (\frac{m_{i+1} - m_i}{\epsilon^2})(x - x_{i+1})^3 - 2(\frac{m_{i+1} - m_i}{\epsilon})(x - x_{i+1})^2 + m_{i+1}(x - x_{i+1}) + y_{i+1}$
has the properties:
$P_{i+1}(x_{i+1}) = y_{i+1}, P_{i+1}(x_{i+1} + \epsilon) = m_i\epsilon + y_{i+1}, P'_{i+1}(x_{i+1} + \epsilon) = m_i, P'_{i+1}(x_{i+1}) = m_{i+1},$

$max_{x \in [x_{i+1}, x_{i+1} + \epsilon]}|P'_{i+1}(x)| = max\{m_i, m_{i+1}\}$ and the graph of $P_{i+1}(x)$ doesn't leave the closed disk $D_\epsilon((x_{i+1}, y_{i+1}))$ for $x \in [x_{i+1}, x_{i+1} + \epsilon]$.

We now modify $h(x)$ to get a function $g(x)$ that agrees with $h(x)$ except at small disks around the corners in the graph of $h(x)$.

First for each $i \in \{2, 3...\}$ choose $\delta_i > 0$ such that the disk $D_{\delta_i}(p_i) \subset W_i$ and $p_j \notin D_{\delta_i}(p_i)$ for $j \neq i$. Let r_i denote that x value where $r_i > x_i$ and $(r_i, h(r_i))$ is the point of intersection of the graph of h with the boundary of $D_{\delta_i}(p_i)$. Let $Q_i(x)$ denote the polynomial

$Q_i(x) = (\frac{t_i - t_{i-1}}{\delta_i^2})(x - x_i)^3 - 2(\frac{t_i - t_{i-1}}{\delta_i})(x - x_i)^2 + t_i(x - x_i) + y_i$

Define $g_i(x) = \begin{cases} Q_j(x), x \in [x_j, r_j], j \leq i \\ h(x) \text{ otherwise} \end{cases}$

Note that $a_j \leq g_j'(x) \leq b_j$ for $x \in [x_j, r_j], j \leq i$.
Now define

$$g(x) = \begin{cases} mx, x \leq 0 \\ g_k(x), x_k \leq x < x_{k-1} \quad k \in \{2, 3, ...\} \\ t_1 x, x \geq x_1 \end{cases}$$

By the Differentiability Lemma, $g'(0) = m$ and g is differentiable on $(-\infty, \infty)$. We now show that $g'(x)$ is continuous on $(-\infty, \infty)$. Note that each $g_i \in C^1$.

Theorem. $g \in C^1$.
Proof. We show that $g_i' \to g'$ uniformly. Let $\epsilon > 0$ be given. Choose $N > 0$ so that for $i > N, max\{|a_i - m|, |b_i - m|\} < \epsilon$. Then for $i > N$ and $x \notin (0, x_i)$, $g_i'(x) = g'(x)$. On the other hand, for $x \in [0, x_i)$, $a_i \leq g_i'(x) \leq b_i$ therefore $|g_i'(x) - m| < \epsilon$ $\to |g_i'(x) - g'(x)| < \epsilon$ for all x for $i > N$.

Question: can this result be extended for limits over C^n functions, where $n > 1$?

[1] Gelbaum, Bernard R. and Olmsted, John. Counterexamples in Analysis, Dover Publications, 2003.

[2] Nanyes, Ollie. Limits of Functions of Two Variables, *College Mathematics Journal*, September 2005, pp.