In this project, we consider logistic models of population growth that have been modified to account for the fact that the carrying capacity of the population may change with time.

Recall that the usual logistic differential equation is:

$P' = kP(1 - \frac{P}{L})$ where L represents the carrying capacity. We will now consider L to be a function of time; that is our first attempt to find a suitable differential equation is:

(1) $P' = kP(1 - \frac{P}{L(t)})$

where $L(t)$ is now a function which gives the variational carrying capacity. We assume that $0 < L(t)$ for all $t > 0$. We can also assume that $0 < P(0) < L(0)$.

Calculus Review Questions

Question One. Why is it reasonable to assume that $P(0) < L(0)$?

Question Two. If L is assumed to be a smooth positive function, given an initial condition $P(0) = P_0$, are we guaranteed to have a unique solution over our region (in the t, P plane) of interest (which is what region?)? Why? Note: there are both existence and uniqueness considerations here.

Question Three. If we assume that $L(t)$ is an increasing function, what can we say about the relative growth rate $\frac{P'}{P}$?

Question Four. If $L(t_1) = P(t_1)$ what can you say about $P'(t_1)$? If $P'(t_1) = 0$, what can you say about $L(t_1)$ and $P(t_1)$?

Question Five. Here is a challenge: if L is a strictly increasing smooth function, conclude that $P(t) < L(t)$ for all $t > 0$ (hint: what can you say about $L'(t)$)?

Question Six. Here is a challenge: If L is a smooth, strictly increasing function, conclude that $P(t) < L(t)$ for all $t > 0$ (hint: what can you say about $L'(t)$)?

Question Seven: Calculate P''' and evaluate at t_1 and leave your answer in terms of L, L' and L''.

Question Eight: Tell how we would get $P^{(n)}$. We can obtain $P^{(n)}$ merely by starting with an expression for $P^{(n-1)}$ and taking the derivative. That is, we differentiate both sides of (1) to obtain P'', then both sides of "$P'' =$" to obtain P''', and so on.

Computer Project

Consider the following candidates for L:

$L_1(t) = L_0 + w_1 t \quad (w_1 > 0)$

$L_2(t) = L_0 + w_2 \ln(1 + t) \quad (w_2 > 0)$

$L_3(t) = L_0 + w_3 t^2 \quad (w_3 > 0)$

Use a computer ODE solver to plot direction fields (of the ODE sans the initial condition) and of solutions to (1) using the appropriate carrying capacity function L_i and given the following table of initial conditions and model parameters:

$P(0)$	$L(0)$	k	w_1	w_2	w_3
4	200	.03	1.4	15	.002
10	200	.03	1.4	15	.002
50	200	.03	1.4	15	.002
4	100	.05	1.4	15	.002
10	100	.05	1.4	15	.002
4	200	.03	2.0	45	.004

Example: for the first row, you would plot the direction fields and solutions to three ODE's:

$P' = .03\,P(1 - \frac{P}{200+1.4t})$, $P' = .03P(1 - \frac{P}{200+15\ln(1+t)})$, $P' = .03P(1 - \frac{P}{200+.002\,t^2})$, each of which has initial condition $P(0) = 4$.

Note: you will be plotting the graphs of the solutions to 6*3 = 18 different ODE's. Don't worry, you won't have to turn in 18 different graphs.

Write a few paragraphs describing the effects that $P(0)$, $L(0)$; k; L_i and w_i have on the solution. Include some carefully chosen graphs as part of your explanation. Note that the first row of data is taken from studies on the population growth of the United States (where P is in millions).

2. Now assume that $L(t) = 200 + 1.4t + 40\sin(\frac{2\pi t}{50})$, and that $k = .05$.

Describe the carrying capacity function. Use the computer to plot a direction field and to plot 3 solutions where $P(0) = 4$; $P(0) = 10$; $P(0) = 50$. Notice that your solution curves for P have local maxima and minima. Comment on what must be happening at such points.