In this project, we consider logistic models of population growth that have been modified to account for the fact that the carrying capacity of the population may change with time.

Recall that the usual logistic differential equation is:

$P' = kP(1 - \frac{P}{L})$ where L represents the carrying capacity. We will now consider L to be a function of time; that is our first attempt to find a suitable differential equation is:

(1) $P' = kP(1 - \frac{P}{L(t)})$

where $L(t)$ is now a function which gives the variational carrying capacity. We assume that $0 < L(t)$ for all $t > 0$. We can also assume that $0 < P(0) < L(0)$.

Question One. Why is it reasonable to assume that $P(0) < L(0)$?

Answer: because it is reasonable to assume that the initial population is less than the assumed starting carrying capacity of the system.

Question Two. If L is assumed to be a smooth positive function, given an initial condition $P(0) = P_0$, are we guaranteed to have a unique solution over our region (in the $t; P$ plane) of interest (which is what region?)? Why? Note: there are both existence and uniqueness considerations here.

Answer: the region of interest is $P > 0; t \geq 0$: If we consider any rectangle R in the first quadrant of the $(t; P)$ plane, we see that $P' = f(t; P) = kP(1 - \frac{P}{L(t)})$ is continuous everywhere (as $L(t) > 0$). Hence a solution exits. The solution is unique as $\frac{\partial f}{\partial P} = k(1 - \frac{P}{L(t)}) + kP(1 - \frac{1}{L(t)})$ is continuous on the whole $(t; P)$ plane.

Question Three. If we assume that $L(t)$ is an increasing function, what can we say about the relative growth rate $\frac{P'}{P}$?

Answer: $\frac{P'}{P} = k(1 - \frac{P}{L(t)})$. Since $k > 0$, as t increases, L increases. When $P(t)$ is relatively small compared to $L(t)$, the relative growth rate is approximately k. Note that $(1 - \frac{P}{L(t)})' = -(\frac{LP' - L'P}{L^2}) = \frac{L'P - LP'}{L^2}$. It is reasonable to assume that initially, $P' > L'$. Otherwise, we are not in a constrained growth situation. Hence $LP' > L'P$ and we expect the relative growth rate to decrease with time. However, the sign of $L'P - P'L$ determines whether the relative growth rate increases, decreases or remains constant.

Question Four. If $L(t_1) = P(t_1)$ what can you say about $P'(t_1)$? If $P'(t_1) = 0$, what can you say about $L(t_1)$ and $P(t_1)$?

Answer: $P' = kP(t)(1 - \frac{P(t)}{L(t)})$. If $P(t_1) = L(t_1)$, $P'(t_1) = 0$. If $P'(t_1) = 0$, then $1 - \frac{P(t_1)}{L(t_1)} = 0$ so $P(t_1) = L(t_1)$.

Question Five. Here is a challenge: if L is a strictly increasing smooth function, conclude that $P(t) < L(t)$ for all $t > 0$ (hint: what can you say about $L'(t)$)?

Answer: suppose t_1 is the first t such that $P(t_1) = L(t_1)$. Let $F(t) = L(t) - P(t)$. $F'(t) = L'(t) - P'(t)$. Because L is strictly increasing, $F'(t_1) = L'(t_1) - P'(t_1) = L'(t_1) > 0$. Because L and P are twice differentiable, they have a continuous derivative; hence there is an interval (a, b) containing t_1 where $F' > 0$. Therefore F is increasing at t_1. But $F(t_1) = 0$; therefore F is negative on (a, t_1) which implies that $P > L$ on (a, t_1), which is impossible.

Note: there is an alternate argument which uses P''.

Question Six. Here is a challenge: If L is a smooth, strictly increasing function, conclude that $P(t) < L(t)$ for all $t > 0$ (hint: what can you say about $L'(t)$)?

Answer to Question 6: $P''(t_1) = P'(t_1)(1 - \frac{P(t_1)}{L(t_1)}) + P(-\frac{L(t_1)P'(t_1) - L'(t_1)P(t_1)}{L^2(t_1)}) = 0 + P(t_1)(-\frac{-L'(t_1)P(t_1)}{(L(t_1))^2}) = L'(t_1)(\frac{P(t_1)}{L(t_1)})^2 = L'(t_1)$ so $P''(t_1) = L'(t_1)$.

Question Seven: Calculate P''' and evaluate at t_1 and leave your answer in terms of L, L' and L''.

Solution: $P = kP(1 - P/L)$; $P' = k(P'(1 - P/L) + P(1 - P/L)')$; $P'' = k(P''(1 - P/L) + 2P'(1 - P/L)' + P(1 - P/L)'')$

$P''' = k(P'''(1 - P/L) + 3P''(1 - P/L)' + 3P'(1 - P/L)'' + P(1 - P/L)''') =$

at $(t = t_1) = k(3P''(1 - P/L)') + P(1 - P/L)''') =$

$k(3P''(\frac{L'P - LP'}{L^2}) + P(\frac{L^2(L''P + L'P' - L'P' - LP'') - 2LL'(L'P - LP')}{L^4})) =$

$k(3P''L'/L) + P/L^4(L^2(L''P - LP'') - 2LL'L'P)) =$

$k(3(L')^2/L + L'' - L' - 2(L')^2/L) =$

$k(L'' - L' - (L')^2/L)$

Question Eight: Tell how we would get $P^{(n)}$.

Answer: We can obtain $P^{(n)}$ merely by starting with an expression for $P^{(n-1)}$ and taking the derivative. That is, we differentiate both sides of (1) to obtain P'', then both sides of "$P'' =$" to obtain P''', and so on.

Computer Project

Consider the following candidates for L:

$L_1(t) = L_0 + w_1 t \quad (w_1 > 0)$

$L_2(t) = L_0 + w_2 \ln(1 + t) \quad (w_2 > 0)$

$L_3(t) = L_0 + w_3 t^2 \quad (w_3 > 0)$

1. Use a computer ODE solver to plot direction fields (of the ODE sans the initial condition) and of solutions to (1) using the appropriate carrying capacity function L_i and given the following table of initial conditions and model parameters:

$P(0)$	$L(0)$	k	w_1	w_2	w_3
4	200	.03	1.4	15	.002
10	200	.03	1.4	15	.002
50	200	.03	1.4	15	.002
4	100	.05	1.4	15	.002
10	100	.05	1.4	15	.002
4	200	.03	2.0	45	.004

Example: for the first row, you would plot the direction fields and solutions to three ODE's:

$P' = .03\,P(1 - \frac{P}{200 + 1.4\,t})$, $P' = .03P(1 - \frac{P}{200 + 15\ln(1+t)})$, $P' = .03P(1 - \frac{P}{200 + .002\,t^2})$, each of which has initial condition $P(0) = 4$.

Note: you will be plotting the graphs of the solutions to 6*3 = 18 different ODE's. Don't worry, you won't have to turn in 18 different graphs.

Write a few paragraphs describing the effects that $P(0)$, $L(0)$; k; L_i and w_i have on the solution. Include some carefully chosen graphs as part of your explanation. Note that the first row of data is taken from studies on the population growth of the United States (where P is in millions).

Answer: in the case where L grows linearly, P is initially concave up and then flattens out to linear growth. Increasing w increases the slope of P, and increasing k causes quicker initial growth of P.

In the case where L grows logarithmically, the graphs of P vary little from the classical constant L solutions. Increasing k increases the initial rate of growth of P and increasing w causes the point of diminishing returns to occur later. The lower values of w makes the graph of P have a "plateau" with a slight upward slope.

In the case where L grows quadratically, the shape of P is greatly affected by w and to a lesser extent by $P(0)$. If w is large, P starts concave up and remains so; one thinks of a quadratic approximation to the exponential function. If w is smaller, we obtain concavity changes; at first P is concave up. The graph switches concavity past the classical "point of diminishing returns" only to become concave up again. The exception to this is at the $P(0) = 50$ case; here the concavity changes are difficult to discern as the graph of P almost looks linear at first.

2. Now assume that $L(t) = 200 + 1{:}4t + 40\sin(\frac{2¼t}{50})$, and that $k = {:}05$. Describe the carrying capacity function. Use the computer to plot a direction …eld and to plot 3 solutions where $P(0) = 4$; $P(0) = 10$; $P(0) = 50$. Notice that your solution curves for P have local maxima and minima. Comment on what must be happening at such points.

One possible answer: each solution starts o¤ increasing and concave up. The $P(0) = 50$ solution then has two slight concavity changes until it reaches a relative maximum. At the relative maximum, $P = L$. But, because P is concave down at that point, say t_a the result of question six implies that $L^0 < 0$; that is L is on a down cycle. Similarly, when P is at a relative minimum, $P = L$ at, say t_b, but L is increasing. Therefore, on $(t_a; t_b)$, L must have passed from decreasing to increasing and have a local minimum at some intermediate point.

The smaller initial value solutions exhibit similar phenomena, and eventually converge to the larger initial value solution. The "change of concavity while P increases" for the $P(0) = 4$ solution is less evident than it is with the other solutions.