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# ON INFINITELY PERIODIC KNOTS

MAT TIMM AND OLLIE NANYES

ABSTRACT. Fox asked the following question: "if for every $g \geq 2$, there is a periodic transformation $T_g$ of period $g$ of the 3-sphere $S^3$ such that $T_g(K) = K$, what kind of knot can $K$ be?" Flapan showed that if $K$ were smooth and if the fixed point set was $S^1$ and disjoint from $K$, that $K$ had to be the unknot. In this paper we show that there are non-trivial wild knots of this type that admit periods of all orders, and that all such knots must have an uncountable number of wild points.

## 1. Introduction

In [3], Fox asked the following question (Question 6, [3]) "if for every $g \geq 2$, there is a periodic transformation $T_g$ of period $g$ of the 3-sphere $S^3$ ($T_g\colon S^3 \to S^3$) such that $T_g(K) = K$ (but not necessarily $T_g(p) = p$ for any $p \in K$), what kind of knot can $K$ be?". Fox then posed Question 7, which asked, "Given a non-trivial knot $K$, which periods $g$ does it permit?" He observed that that the answer may depend on the fixed point set $F$ of $T_g$, whether $T$ is orientation reversing or preserving, and on $K \cap F$.

A knot $K$ is said to be infinitely periodic if, for each $g \in N$, there exists a periodic homeomorphism $T_g : S^3 \to S^3$ such that $T_g(K) = K$. The fixed point set of a homeomorphism $T : S^3 \to S^3$ will be denoted by $F = F(T) = \{p \in S^3 : T(p) = p\}$. Note that for any such $T$, $F(T)$ is homeomorphic to one of $\emptyset$, $S^0, S^1$, $S^2$ or $S^3$ [3]. We will adopt Flapan's notation [2]: a knot $K$ will be said to be $(a, b)$-periodic if there is a $T$ with $F(T)$ homeomorphic to $a$ such that $F(T) \cap K$ is homeomorphic to $b$. $K$ will be said to be infinitely $(a, b)$ periodic if for each $g \geq 2$ there is a $T$ of type $(a, b)$ with period $g$. In our paper, we will be interested in infinitely $(S^1, \emptyset)$ periodic knots.

In [7], Seifert showed that any smooth torus knot is infinitely periodic with $F = \emptyset$. In [2], Flapan showed that the torus knots are the only smooth, infinitely periodic knots $K$ with $F = \emptyset$ and that the only smooth $(S^1, \emptyset)$ periodic knot which admits infinitely many periods is the unknot.

Our main result shows that the smooth condition on $K$ is essential to Flapan's argument; we will show how to construct an infinitely $(S^1, \emptyset)$ periodic wild knot (Theorem (2.1)). These knots are contained in certain solid tori, and their complements in these solid tori are connected open 3-manifolds with one boundary component that have non-trivial cyclic self-covers of all orders. In section 3 we show that there are an uncountable number of inequivalent $(S^1, \emptyset)$ periodic wild

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knots. We will also show that all such knots must have an uncountable number of wild points (Theorem (4.1).

The method used to construct the knots of interest in fact illustrates a general method that can be used to construct many spaces with cyclic self-covers of all orders. For related work, see [1].

## 2. Construction of an infinitely periodic knot

Henceforth, when we say that a knot $K$ is infinitely periodic we mean infinitely $(S^1, \emptyset)$ periodic. Note that we do not require that each $T_g$ have the same fixed point set $F$ for all $g$.

We start our construction of an infinitely periodic knot $K$ as follows: view $S^3$ as $R^3 \cup \{\infty\}$. We will use cylindrical coordinates for $R^3 = \{(r, \theta, z) : r \geq 0, 0 \leq \theta < 2\pi, z \in R\}$. Consider a solid torus $V \subset R^3$, which we will parametrize as $S^1 \times D^2 = \{(\theta, \rho, \phi) : 0 \leq \theta < 2\pi, 0 \leq \rho \leq 2, 0 \leq \phi < 2\pi\}$. $(\rho, \phi)$ represents a polar coordinate system of a meridional disk of $V$; the units used for $\rho$ are not the standard distance units in $R^3$. For example, $(0, 2, \pi) \in V$ has $R^3$ coordinates $(\frac{3}{2}, 0, 0)$ and $(\frac{\pi}{2}, 1, \frac{3\pi}{4}) \in V$ has $R^3$ coordinates $(\frac{2-\sqrt{2}}{2}, \frac{\pi}{2}, \frac{\sqrt{2}}{2})$. The centerline of $V$ will be identified with the unit circle in the $z = 0$ plane of $R^3$, with $\theta$ being used for both $S^1 \subset V$ and $R^3$. $V \cap \{z = 0\}$ will be identified with the annulus $\{(r, \theta, 0) : \frac{1}{2} \leq r \leq \frac{3}{2}, 0 \leq \theta < 2\pi\}$.

Let $C$ denote the standard "middle thirds" Cantor set in the interval $[0, 1]$, and $C^*$ its image under $\pi : [0, 2] \to V$, $\pi(x) = \{(\pi x, 0, -)\}$. Note that $\theta = \pi x$. Let $D$ denote the set of intervals that are deleted from $[0, 1]$ to form $C$, together with $(1, 2)$. That is,

$$D = \{(1,2)\} \cup \{(\frac{1}{3}, \frac{2}{3}), (\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9}), (\frac{1}{27}, \frac{2}{27}), (\frac{7}{27}, \frac{8}{27}), (\frac{19}{27}, \frac{20}{27}), (\frac{25}{27}, \frac{26}{27}).....\}.$$

Let $D_{i,j} \in D$ be defined to be the $j$'th deleted interval of length $\frac{1}{3^i}$. We let $D_{0,0}$ denote the open interval $(1, 2)$. Let $D^*_{i,j}$ denote the image of $D_{i,j}$ under $\pi$ and $D^*_{0,0}$ denote the semi-circular segment $((\pi, 0, -), (2\pi, 0, -))$ and $D^*$ denote the collection of the $D^*_{i,j}$. For each $D_{i,j} = (\frac{p}{3^i}, \frac{p+1}{3^i})$ ($i \geq 0, j \geq 0$, where $j \leq 2^{i-1}$ for $i \geq 1$), associate an open "curvilinear double cone" $A_{i,j} \subset V$ (see Figure 1), where

$A_{i,j} = \{(\pi x, \rho, \phi) | \frac{p}{3^i} < x < \frac{p+1}{3^i}, 0 \leq \rho < a(x)$ and $a(x) = \frac{1}{2}(min\{x - \frac{p}{3^i}, \frac{p+1}{3^i} - x\})\}$.

We will let $C_{i,j}$ denote the open "curvilinear double cone" $C_{i,j} = \{(\pi x, \rho, \phi) | \frac{p}{3^i} < x < \frac{p+1}{3^i}, 0 \leq \rho < b(x)$ and $b(x) = \frac{3}{2}(min\{x - \frac{p}{3^i}, \frac{p+1}{3^i} - x\})\}$. We are using this "nested cone" construction to facilitate our contstruction of the periodic homeomorphisms $T$; in particular the region between the two cones will serve as a "buffer" for the expansion/contraction part of each $T$.

Consider a ball pair $(B, k)$, where $B$ is a 3-ball and $k$ is a properly embedded arc (possibly wild at its endpoints) with $k \cap \partial B = \{c\} \cup \{d\}$. Assume that $k$ is knotted in $B$ (rel $(c, d)$). That is, $k$ together with an arc in $\partial B$ is a non-trivial knot in $B \subset S^3$. Choose a specific knot type for $k$ (e.g., $k$ could be the trefoil as indicated in Figure 1, or the reader's favorite knot). We call $k$ a *pattern arc* for $K$. For each $A_{i,j}$ there is a smooth embedding $\psi_{i,j} : (B, k) \to V$ where $\psi_{i,j}(c) = \pi(\frac{p}{3^i})$, $\psi_{i,j}(d) = \pi(\frac{p+1}{3^i})$, $\psi_{i,j}(\int(B, k)) = A_{i,j}$ and, for all $i, j$, all of the

$\psi_{i,j}(B,k) \subset A_{i,j}$ are homeomorphic by orientation preserving homeomorphisms. That is, we glue in the "same" $k$ into each $A_{i,j}$

Let $K$ be the simple closed curve which consists of $C^* \cup \{\cup_{i,j}\{\psi_{i,j}(k)\}, (i \geq 0, j \geq 0$, where $j \leq 2^{i-1}$for $i \geq 1)\}$.

That is, $K$ is the knot formed by replacing the deleted intervals with copies of the arc $k$, where the copies of $k$ are properly embedded in the $C_{i,j}$. Clearly, $K$ is a wild, non-trivial knot. Figure 1 shows some of the stages of $K$.

Theorem (2.1). *$K$ is infinitely $(S^1, \emptyset)$-periodic.*

*Proof.* First note that for each $n \in \{1, 2, 3, \dots\}$ there exists a homeomorphism $f_n$ of the centerline circle $S^1 \subset V$ (recall $S^1 = \{(\theta, 0, 0) \in V, \theta \in [0, 2\pi)\}$) of period $n$ that maps $D^*$ to itself. We describe this map as follows:

$$f_1(\theta) = \theta$$

$$f_2(\theta) = \begin{cases} \theta + \frac{2\pi}{3}, & \theta \in [0, \frac{\pi}{3}] \\ 3\theta (\operatorname{mod} 2\pi), & \theta \in (\frac{\pi}{3}, \frac{2\pi}{3}] \\ (\theta + \frac{4\pi}{3})(\operatorname{mod} 2\pi), & \theta \in (\frac{2\pi}{3}, \pi] \\ \frac{1}{3}\theta, & \theta \in (\pi, 2\pi) \end{cases}$$

$$f_3(\theta) = \begin{cases} \theta + \frac{2\pi}{3^2}, & \theta \in [0, \frac{\pi}{3^2}] \\ 3\theta (\operatorname{mod} 2\pi), & \theta \in (\frac{\pi}{3^2}, \frac{2\pi}{3}] \\ \frac{(\theta + \frac{4\pi}{3})(\operatorname{mod} 2\pi)}{3}, & \theta \in (\frac{2\pi}{3}, \pi] \\ \frac{1}{3^2}\theta, & \theta \in (\pi, 2\pi) \end{cases}$$

$$f_n(\theta) = \begin{cases} \theta + \frac{2\pi}{3^{(n-1)}}, & \theta \in [0, \frac{\pi}{3^{(n-1)}}] \\ 3\theta (\operatorname{mod} 2\pi), & \theta \in (\frac{\pi}{3^{(n-1)}}, \frac{2\pi}{3}] \\ \frac{(\theta + \frac{4\pi}{3})(\operatorname{mod} 2\pi)}{3^{n-2}}, & \theta \in (\frac{2\pi}{3}, \pi] \\ \frac{1}{3^{n-1}}\theta, & \theta \in (\pi, 2\pi) \end{cases}$$

It will be helpful to define a function

$$\kappa : [0, 2\pi) \times Z^+ \to \{1, 3, 3^{-1}, 3^{-2}, ...3^{-m}, ..\}$$

by

$$\kappa(\theta, n) = \begin{cases} 1, & \theta \in [0, \frac{\pi}{3^{(n-1)}}] \\ 3, & \theta \in (\frac{\pi}{3^{(n-1)}}, \frac{2\pi}{3}] \\ \frac{1}{3^{n-2}}, & \theta \in (\frac{2\pi}{3}, \pi] \\ \frac{1}{3^{n-1}}, & \theta \in (\pi, 2\pi) \end{cases} \qquad (n > 1)$$

$\kappa(\theta, n) = 1$ for $n = 1$.

Next, we can define a homeomorphism $g_{(a,b,\kappa)}$ of the interval $[0, 2]$ (where $0 < a < b < 2, \kappa a < b$) which takes $[0, a]$ to $[0, \kappa a]$ and is the identity on $[b, 2]$ as follows:

$$g_{(a,b,\kappa)}(\rho) = \begin{cases} \kappa\rho, & 0 \leq \rho \leq a \\ \rho(\frac{(1-\kappa)\rho + \kappa b - a}{b-a}), & a < \rho < b \\ \rho, & b \leq \rho \leq 2\,. \end{cases}$$

Now, we define a homeomorphism $T_{*n}$ of period $n$ on $V$ which maps $D^* \subset S^1 = \{(\theta, 0, -)\}$ to itself:

$$T_{*n}(\theta,\,\rho,\phi) = \begin{cases} (f_n(\theta), g_{(a(x),b(x),\kappa(\theta,n))}(\rho), \phi); & \theta \in D^* \\ (f_n(\theta), \rho, \phi); & \theta \notin D^* \end{cases} \quad \text{(recall: } \theta = \pi x)$$

The effect of $T_{*n}$ for $n \geq 2$ is to perform an appropriate expansion or contraction in the $x$ coordinate, and expansion or squeezing in the $\rho$ coordinate. Note that for $(\theta, \rho, \phi) \in \partial V$, $T_{*n}(\theta, \rho, \phi) = (f_n(\theta),\, \rho, \phi)$. Hence it is easy to extend $T_{*n}$ to a homeomorphism $T_n$ of $R^3$which has fixed point set $F = \{(0, -, z)|z \in R\}$ and has period $n$ where $T_n|_V = T_{*n}$. Note that $T_n(K) = K$ setwise. Theorem (2.1) is proved. □

## 3. Construction of an uncountable number of mutually inequivalent $(S^1, \emptyset)$-periodic knots

In [5], McPherson shows how to construct an uncountable number of mutually inequivalent Fox-Artin arcs which have one wild endpoint of penetration index three. We will use these arcs as our pattern arc $k$ to demonstrate that there are an uncountable number of mutually inequivalent $(S^1, \emptyset)$-periodic knots.

Let $k$ be an embedded arc in $S^3$ with endpoints $p, q$. Assume that $k$ is tame at all of its points except for possibly $p$. Let $E_1, E_2, \ldots$ be a system of tame closed 3-balls where $\cap_{i=1}^{\infty} E_i = p$ such that for all $i \geq 1, E_{i+1} \subset \int(E_i)$ and $k \cap (E_i - \int(E_{i+1}))$ consists of exactly three arcs: $\alpha_i$ which runs between $\partial E_i$ and $\partial E_{i+1}, \beta_i$ which runs between two points of $\partial E_i$, and $\gamma_i$ which runs between two points of $\partial E_{i+1}$. If $\beta_i$ and $\gamma_i$ are unsplittable in each $E_i - \int(E_{i+1})$ (in the sense that if one turns $\beta_i$ and $\gamma_i$ into closed loops $\overline{\beta}_i, \overline{\gamma}_i$ by adding arcs along $\partial E_i$ and $\partial E_{i+1}$ respectively then $\overline{\beta}_i \cup \overline{\gamma}_i$ is an unsplittable link in $S^3$) then $k$ is said to be a *Fox-Artin arc of penetration index 3* (the penetration index comes from the three arcs in each $E_i - \int(E_{i+1})$). Figure 2 shows an example of a Fox-Artin arc. In [4] McPherson shows that all Fox-Artin arcs are wild (and therefore non-trivial) and in [5] he shows that there are an uncountable number of Fox-Artin arcs of penetration index 3 which have inequivalent "local types" at wild point $p$. It follows that there are an uncountable number of mutually inequivalent Fox-Artin arcs.

THEOREM (3.1). *Let $K_1$ and $K_2$ be two knots which are constructed in the manner of Section 2, with pattern arcs $k_1$ and $k_2$ respectively. If $K_1$ is equivalent to $K_2$ then $k_1$ is equivalent to $k_2$.*

THEOREM (3.2). *There exists an uncountable number of mutually inequivalent $(S^1, \emptyset)$-periodic knots.*

*Proof of Theorem (3.2).* Follows directly from Theorem (3.1). □

*Proof of Theorem (3.1).* First some notation (refer to Section 2): $C_i^*$ refers to the image of the Cantor set in knot $K_i$, $k_{m,n}^i$ refers to $\psi_{m,n}(k_i)$ in $K_i$, $p^i$ is the "wild endpoint" of $k_i, q^i$ the tame endpoint of $k_i$, $p_{m,n}^i$ is $\psi_{m,n}(p_i)$ and $q_{m,n}^i$ is $\psi_{m,n}(q_i)$. Let $h : S^3 \to S^3$ be a homeomorphism such that $h(K_1) = K_2$. We need a technical lemma.

LEMMA (3.3). *For all* $m, n \in \{0, 1, 2, \dots\}$, $h(k^1_{m,n}) = k^2_{r,s}$ *for some* $r, s \in \{0, 1, \dots\}$.

*Proof of the lemma.* For each $q^1_{m,n}$ there exists a disk $Q_{m,n}$ with a product neighborhood $[-1, 1] \times Q_{m,n}$ such that $(\{0\} \times Q_{m,n}) \cap K_1 = q^1_{m,n}$ and $([0, 1] \times Q_{m,n}) \cap k^1_{m,n}$ consists of tame arcs (tamely embedded in $([0, 1] \times Q_{m,n})$), one of which has $q^1_{m,n}$ as an endpoint. We say that these $q^1_{m,n}$ are *tame from one side*. Similarly, for each $p^1_{m,n}$ there exists a disk $P_{m,n}$ with a product neighborhood $[-1, 1] \times P_{m,n}$ such that $(\{0\} \times P_{m,n}) \cap K_1 = p^1_{m,n}$ and $([-1, 0) \times P_{m,n}) \cap k^1_{m,n}$ consists of tame open arcs (tamely embedded in $([-1, 0) \times P_{m,n})$), one of which has $p^1_{m,n}$ as an endpoint (and, of course, is wild when $p^1_{m,n}$ is added in). We say that these $p^1_{m,n}$ are *almost tame from one side*.

If $x \in K^1$ is (almost) tame from one side, then $h(x)$ is (almost) tame from one side. If $y \in C^*_2$ then every open neighborhood contains wild points of $K^2$ on both sides of $y$. Therefore $y \neq h(x)$, which implies that $y$ cannot be the image of any point of any $k^1_{m,n}$ because all of the points of $k^1_{m,n}$ are either tame, almost tame from one side, or tame from one side. Therefore $h(q^1_{m,n}) = q^2_{r,s}$ and $h(p^1_{m,n}) = p^2_{u,v}$. If $r \neq u$ or $s \neq v$, any subarc of $K^2$ running from $q^2_{r,s}$ to $p^2_{u,v}$ must pass through an infinite number of wild points. Therefore $(r, s) = (u, v)$. Hence $h$ takes $k^1_{m,n}$ to $k^2_{u,v}$ which implies that $k^1$ and $k^2$ are equivalent. Lemma (3.3) and Theorem (3.1) are proved. □

## 4. Characterization of infinitely $(S^1, \emptyset)$-periodic knots

We now give a necessary condition for a knot $K$ to be infinitely $(S^1, \emptyset)$-periodic.

THEOREM (4.1). *If* $K$ *is a non-trivial infinitely* $(S^1, \emptyset)$*-periodic knot, then* $K$ *has an uncountable number of wild points.*

The proof of Theorem (4.1) will follow after some lemmas and propositions. First, we establish some notation. If $A$ is a set, $A'$ will denote the limit points of $A$. A point $x \in K$ is said to be tame if there exists a closed p.l. 3-ball $B$ such that $x \in \int(B)$ and $(B,\, B \cap K)$ is a standard ball pair. If $x$ is not a tame point of $K$, then $x$ is called wild. We denote the set of wild points of $K$ by $W$. Note that Flapan's work implies that, for the infinitely periodic knots in which we are interested, $W \neq \emptyset$. Also note that $W$ is a compact set in the standard subspace topology. We can start by assuming that $K$ has tame points, else the theorem follows trivially.

For each $p \in N$, let $T_p : S^3 \to S^3$ be a given fixed periodic homeomorphism of period $p$ acting freely in $K$. Let $S = \{T_p : p \in N\}$. By $T^k_p$ we mean the composition of $k$ copies of $T_p$. Of course, $T^p_p$ is the identity map. We let $G$ be the group generated by $S$. Note that it consists of all finite compositions of the homeomorphisms $T^k_p$. We denote the orbit of $x \in K$ by $O(x) = \{y | \, y = T(x), T \in G\}$.

PROPOSITION (4.2). *For all* $x \in K$, $O(x)$ *is infinite.*

*Proof.* Suppose there is an $x \in K$ with $O(x)$ a finite set. Say, $O(x) = \{x = x_1, \dots, x_k\}$. Let $p = k! + 1$. Let $T_p \in S$ be the given homeomorphism of period $p$. We see that $T_p$ permutes the points of $O(x)$ without fixed points since $F \cap K = \emptyset$. That is, the restriction $(T_p|O(x))$ can be thought of as a permutation $\sigma \in Sym(k)$, the group of permutations of $k$ symbols. So the order of $\sigma$ divides $k!$. Thus $\sigma^{k!} = (T_p|O(x))^{k!} = T_p^{k!}|O(x)$, which is the identity in $O(x)$. But then $T_p^{k!}$ would be the identity, which contradicts the fact that the order of $T_p$ is $k!+1$. The proposition is proved. □

Since homeomorphisms take wild points to wild points, if $x \in W$, then $O(x) \subset W$. Thus we have now established that the set of wild points is at least countably infinite. Suppose $x \in W$. Since $W$ is compact and $O(x) \subset W$ is infinite, $O(x)' \subset W$ is not empty. We will establish that we can assume, with no loss of generality, that $O(x)$ contains none of its limit points.

LEMMA (4.3). *If $x \in W$ and if $O(x) \cap (O(x))' \neq \emptyset$, then $W$ is uncountable.*

*Proof.* Suppose there exists $y \in O(x) \cap (O(x))'$. Then there is some $T \in G$ such that $y = T(x)$. Suppose $U$ is open and $x \in U$. Then the open set $T(U)$ contains an infinite number of points of $O(x)$. Therefore, since $T$ is a homeomorphism, $U$ also contains an infinite number of points of $O(x)$. Hence $O(x) \subset O(x)'$. Hence $\overline{O(x)} \subset O(x)' \subset (\overline{O(x)})'$. That is, all points of the closure of $O(x)$ are limit points of $\overline{O(x)}$. Since $\overline{O(x)}$ is compact, $\overline{O(x)}$ is uncountable (see, e.g., Theorem 6.5, page 176 of [6]). But $\overline{O(x)} \subset W$, therefore $W$ is uncountable. The lemma is proved. □

We need one more lemma:

LEMMA (4.4). *If $y_a \in O(y)'$, then $O(y_a)' \subset O(y)'$.*

*Proof.* It suffices to show that $O(y_a) \subset O(y)'$. Let $x \in O(y_a)$ and $U$ be an open set which contains $x$. Then $x = T(y_a)$ for some $T \in G$. $T^{-1}(U)$ is open and contains $y_a$. Hence $T^{-1}(U)$ contains an infinite number of points of $O(y)$. Therefore $U$ contains an infinite number of points of $O(y)$. The lemma is proved. □

*Proof of Theorem (4.1).* We will prove Theorem (4.1) by showing that no one to one map from the integers to $W$ can be onto. From Lemma (4.3), we will assume that $O(x)$ contains none of its limit points. Let $f : N \to W$ be a one to one map and let $f(n) = y_n$. We will use induction to show the following: given subsets $V_1, V_2, \ldots, V_k$ which are open in $K$, have tame frontier (frontier in $K$) and contain $y_1, y_2, \ldots, y_k$, respectively, where for all $1 \leq i < j \leq k$, either $\overline{V}_i \cap \overline{V_j} = \emptyset$ or $V_i = V_j$, we can find another open set $V_{k+1}$ containing $y_{k+1}$ such that $V_{k+1} = V_j$ for some $1 \leq j \leq k$ or $\overline{V_{k+1}} \cap \overline{V}_j = \emptyset$ for all $1 \leq j \leq k$. Furthermore $K - (V_1 \cup V_2 \cup \ldots V_k \cup V_{k+1})$ contains an infinite number of wild points, namely $O(y_p)'$, for some $p \leq k+1$.

Consider $y_1$. By Lemma (4.3), there is an open set $V_1 \subset K$ which separates $y_1$ from all other points of $O(y_1)$. It follows that $V_1$ contains no limit points of $O(y_1)$. We can assume that $O(y_1)'$ is countable, since, if $O(y_1)'$ were uncountable, $O(y_1) \subset W$ and $W$ is closed; it follows that $O(y_1)'$ would be an uncountable subset of $W$, which would prove the theorem. Note that we can assume with no loss of generality that $\mathrm{Fr}(V_1) \subset K$ is tame. For, if $\mathrm{Fr}(V_1)$ were wild, we could attempt to find a smaller open interval (open in $K$) which contains $y_1$

whose endpoints are tame; if such an interval cannot be found then $W$ must be uncountable. Let $M_1 = K - V_1$. $M_1$ is a compact set which contains $O(y_1)'$.

Now we proceed by induction. Assume by hypothesis of induction that we have open sets $V_1, V_2, \ldots, V_k$ containing $y_1, y_2, \ldots, y_k$ respectively, and for all $1 \leq i < j \leq k$, either $\overline{V}_i \cap \overline{V}_j = \emptyset$ or $V_i = V_j$ and $\mathrm{Fr}(V_i)$ is tame. We also have $O(y_p)' \subset K - (V_1 \cup V_2 \cup \cdots \cup V_k) = M_k$ for some $p$, $1 \leq p \leq k$. Consider $y_{k+1}$. If $y_{k+1} \in \overline{V}_1 \cup \overline{V}_2 \cup \cdots \cup \overline{V}_k$, say, $y_{k+1} \in \overline{V}_r$, then $y_{k+1} \in V_r$ (recall: $\mathrm{Fr}(V_r)$ is tame). Set $V_{k+1} = V_r$.

Otherwise, $y_{k+1} \notin \overline{V}_1 \cup \overline{V}_2 \cup \cdots \cup \overline{V}_k$ and $\mathrm{Fr}(V_i)$ is tame for each $1 \leq i \leq k$. There are two cases to consider.

**Case 1**. If $y_{k+1} \notin O(y_p)'$ then $y_{k+1} \in K - (O(y_p)' \cup (\overline{V_1} \cup \overline{V_2} \cup \cdots \cup \overline{V_k}))$ which is an open set in $K$. Use regularity to find an open $V_{k+1}$such that $y_{k+1} \in V_{k+1}$ and $\overline{V}_{k+1} \cap (O(y_p)' \cup (\overline{V_1} \cup \overline{V_2} \cup \cdots \cup \overline{V_k})) = \emptyset$. We can assume with no loss of generality that $\mathrm{Fr}(V_{k+1})$ is tame. Furthermore, we can define $M_{k+1} = K - (V_1 \cup V_2 \cup \cdots \cup V_{k+1})$. Note that $M_{k+1}$ is compact and contains $O(y_p)'$.

**Case 2**. If $y_{k+1} \in O(y_p)'$, then by Lemma (4.4) $O(y_{k+1})' \subset O(y_p)'$ which implies that $O(y_{k+1})' \subset O(y_p)' \subset K - (V_1 \cup V_2 \cup \cdots \cup V_k)$. But, since $y_{k+1} \notin O(y_{k+1})', y_{k+1}$ belongs to the open set $K - (O(y_{k+1})' \cup (\overline{V_1} \cup \overline{V_2} \cup \cdots \cup \overline{V_k}))$. As in case 1, this open set contains an open neighborhood $V_{k+1}$ of $y_{k+1}$ in $K$ such that $\overline{V}_{k+1} \cap \overline{V}_i = \emptyset$ for all $1 \leq i \leq k$. Again we can assume that $\mathrm{Fr}(V_{k+1})$ is tame and that the compact set $M_{k+1} = K - (V_1 \cup V_2 \cup \cdots \cup V_{k+1})$ equals $(K - (V_1 \cup \cdots \cup V_k)) \cap (K - V_{k+1}))$ which contains $O(y_{k+1})'$.

Therefore we have obtained $M_{k+1} = K - (V_1 \cup V_2 \cup \cdots \cup V_k \cup V_{k+1})$ and have shown that $M_{k+1} \cap W$ is nonempty.

Now consider the nested compact sets $(M_1 \cap W) \supset (M_2 \cap W) \supset \ldots (M_k \cap W) \supset \ldots$. These sets are all compact and the collection $\{M_k \cap W \mid k \in N\}$ has the finite intersection property. Hence $\cap_{i=1}^{\infty}(M_i \cap W) \neq \emptyset$ and contains no point of the sequence $\{y_i\}$. Hence $\cap_{i=1}^{\infty}(M_i \cap W)$ is not in the range of $f$. Therefore, the set $W$ must be uncountable. Therefore Theorem (4.1) is proved. ☐

## 5. Questions

Note that our example of an infinitely periodic knot is non-prime. Are there any prime non-trivial infinitely periodic knots? If so, are all such examples wild at every point? Does the wild point set of an infinitely periodic knot always contain a Cantor set?

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Bradley University
Peoria, IL 61625
USA
onanyes@hilltop.bradley.edu
mtimm@bradley.edu

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