1. Find the area enclosed by one loop of the curve: \( r = 4 \sin(3\theta) \)

Solution: bounds: \( \sin(x) \) cycles from 0 to 1 and back to 0 as \( x \) goes from 0 to \( \pi \).

Let \( x = 3\theta \rightarrow \frac{1}{3} = \theta \rightarrow \) we get a loop as \( \theta \) goes from 0 to \( \frac{\pi}{3} \)

\[
A = \frac{1}{2} \int_{0}^{\frac{\pi}{3}} r^2 d\theta = \frac{1}{2} \int_{0}^{\frac{\pi}{3}} (4 \sin(3\theta))^2 d\theta = \frac{8}{3} \int_{0}^{\frac{\pi}{3}} (\sin^2(3\theta))d\theta = 4 \int_{0}^{\frac{\pi}{3}} 1 - \cos(6\theta)d\theta = 4 \left[ \frac{\theta}{2} - \frac{1}{6} \sin(6\theta) \right]_{0}^{\frac{\pi}{3}} = \frac{4}{3} \pi
\]

2. Write the equation: \( z = x^2 + y^2 \) in cylindrical coordinates and simplify as much as possible.

\( x^2 + y^2 = r^2 \) so this is just \( z = r^2 \).

3. Write the equation \( 1 = 2 \cos(\phi) \) in rectangular (cartesian) coordinates and simplify as much as possible. What kind of surface is this?

To see what kind of surface this is, note that \( \phi = \frac{\pi}{3} \); this gives a cone. Even if you didn’t see this, you’d get

\[
x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi; \text{ note that } \frac{1}{r} = \cos \phi \rightarrow \sin \phi = \sqrt{1 - \cos^2 \phi} = \frac{\sqrt{3}}{2}
\]

So \( x = \rho \frac{\sqrt{3}}{2} \cos \theta, y = \rho \frac{\sqrt{3}}{2} \sin \theta, z = \frac{1}{r} \rho \rightarrow x^2 + y^2 = \frac{3}{4} \rho^2 = 3z^2
\]

\( x^2 + y^2 = 3z^2 \)

4-6. Given \( \vec{T}(t) = < te^t, te^{-t}, t > \)

4. Find a formula for the unit tangent vector \( \vec{T}' \).

\[
\vec{T}'(t) = < te^t + e^t, e^{-t} - te^{-t}, 1 >, \| \vec{T}'(t) \| = \sqrt{(te^t + e^t)^2 + (e^{-t} - te^{-t})^2 + 1}
\]

\[
\vec{T} = \frac{< te^t + e^t, e^{-t} - te^{-t}, 1 >}{\sqrt{(te^t + e^t)^2 + (e^{-t} - te^{-t})^2 + 1}}
\]

5. Find an equation for the line tangent to \( \vec{T}(t) \) at \( t = 1 \).

\( \vec{T}'(1) = < e + e, e^{-1} - e^{-1}, 1 >= < 2e, 0, 1 >; \vec{T}(1) = < e, e^{-1}, 1 > \)

The tangent line equation is \( \vec{S}(u) = < e, e^{-1}, 1 > + u < 2e, 0, 1 > \)

6. Find an equation for the normal plane at \( t = 1 \).

The normal plane has \( \vec{T}' \) as a normal vector. So using \( \vec{T}'(1) = < 2e, 0, 1 > \) from the previous problem, we get

\[
< 2e, 0, 1 > \cdot < x - e, y - e^{-1}, z - 1 >= 0 \rightarrow \ 2e(x - e) + z - 1 = 0
\]
7. Given \( \vec{r}'(t) = < 2 \cos(\frac{\pi}{2} t), 4 \sin(\frac{\pi}{2} t), t > \) find the curvature of \( \vec{r}'(t) \) at the point (0, 4, 1)
\( \vec{r}' = <-\pi \sin(\frac{\pi}{2} t), 2\pi \cos(\frac{\pi}{2} t), 1 >, r'' = <-\frac{\pi^2}{2} \cos(\frac{\pi}{2} t), -\pi^2 \sin(\frac{\pi}{2} t), 0 >; \) at \( t = 1 \) these become:

\[
\vec{r}' = <-\pi, 0, 1 >, r'' = < 0, -\pi^2, 0 >
\]
\[
||r'|| = \sqrt{\pi^2 \sin^2(\frac{\pi}{2} t) + 4\pi^2 \cos^2(\frac{\pi}{2} t) + 1} = \sqrt{\pi^2 + 3\pi^2 \cos^2(\frac{\pi}{2} t) + 1}; \text{ at } t = 1 \text{ this is } \sqrt{\pi^2 + 1}
\]

\[
\vec{r}' \times r'' = \det \begin{bmatrix}
\vec{i} & \vec{j} & \vec{k} \\
-\pi & 0 & 1 \\
0 & -\pi^2 & 0
\end{bmatrix} = \langle \pi^2, 0, \pi^3 > = \pi^2 < 1, 0, \pi >
\]

The curvature \( \kappa = \frac{||r' \times r''||}{||r'||^3} = \frac{\pi^2 \sqrt{1 + \pi^2}}{(\sqrt{\pi^2 + 1})^3} = \frac{\pi^2}{\pi^2 + 1} \)

8-11. Suppose that \( \vec{r} \) passes through the point (3, -1, 2) when \( t = 0 \) and has unit tangent vector
\( \vec{T} = \frac{1}{\sqrt{11}} < \sin(t) + \cos(t), \cos(t) - \sin(t), 3 > \)

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9. Find \( \vec{N} \) (a formula which is valid for any time \( t \))

\[
\vec{N} = \frac{\vec{T'}}{||\vec{T}'||}; \vec{N'} = \frac{1}{\sqrt{11}} < \cos(t) - \sin(t), -\sin(t) - \cos(t), 0 > \Rightarrow \vec{N} = \frac{1}{\sqrt{11}} < \cos(t) - \sin(t), -\cos(t) + \sin(t), 3 >
\]

9. Find \( \vec{B} \) at \( t = 0 \).

Note: \( \vec{T}(0) = \frac{1}{\sqrt{11}} < 1, 1, 3 >, \vec{N}(0) = \frac{1}{\sqrt{12}} < 1, -1, 0 > \)

\[
\vec{B} = \vec{T} \times \vec{N} = \frac{1}{\sqrt{11}} \frac{1}{\sqrt{12}} \det \begin{bmatrix} i & j & k \\ 1 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \frac{1}{\sqrt{122}} < 3, 3, -2 >
\]

10. Find an equation for the osculating plane when \( t = 0 \).
Use \( \vec{B} \) as a normal vector (actually, \( \sqrt{22} \vec{B} \)): \( < 3, 3, -2 > \cdot < x - 3, y - 1, z - 2 > = 0 \)

\[
3(x - 3) + 3(y + 1) - 2(z - 2) = 0 \text{ is an equation.}
\]

11. Find a formula for \( \vec{r}'(t) \) which is valid at any time \( t \). Hint: integration.
\( \vec{r}'(t) = \int \vec{r}'(t)dt = \int \sqrt{11} \vec{T}dt = \sqrt{11} < -\cos(t) + \sin(t), \sin(t) + \cos(t), 3t > + \vec{C} \)

Recall \( \vec{r}'(0) = < 3, -1, 2 > \Rightarrow \vec{N} = \sqrt{11} < -\cos(0) + \sin(0), \sin(0) + \cos(0), 3 * 0 > + \vec{C} \)
\[ <3, -1, 2> = \sqrt{11} < -1, 1, 0 > + \vec{C} \rightarrow \vec{C} = <3 + \sqrt{11}, -1 - \sqrt{11}, 2 > \]

So \( \vec{r}(t) = <\sqrt{11} (\sin(t) - \cos(t)) + 3 + \sqrt{11}, \sqrt{11} (\sin(t) + \cos(t)) - 1 - \sqrt{11}, 3\sqrt{11} t + 2 > \]