Math 223, Spring 2004, 9 am Section.
Calculator is allowed; show your work.

1. Given the polar curve: \( r = 1 + 2 \cos(\theta) \), find the area enclosed by the inner loop. Give an exact answer (e. g., no decimal answers) and show your work (e. g., set up your integral, show how you obtained your limits).

Limits for inner loop:

\[ r = 0 \to 1 + 2 \cos(\theta) = 0 \to \cos(\theta) = -\frac{1}{2} \to \theta = \frac{2}{3} \pi, \theta = \frac{4}{3} \pi \]

(2 pts) \( \frac{1}{2} \int_{a}^{b} r^2 d\theta = \)

(2 pts) \( \frac{1}{2} \int_{\frac{2}{3} \pi}^{\frac{4}{3} \pi} (1 + 2 \cos(\theta))^2 d\theta = \) note: if you use the calculator for the integral from this point onward, you can get full credit if the final answer is correct. To get more partial credit from this point, you must show work as follows.

\( \frac{1}{2} \int_{\frac{2}{3} \pi}^{\frac{4}{3} \pi} 1 + 4 \cos(\theta) + 4 \cos^2(\theta) d\theta = \)

\( \frac{1}{2} \int_{\frac{2}{3} \pi}^{\frac{4}{3} \pi} 1 + 4 \cos(\theta) + 2(1 + \cos(2\theta)) d\theta = \)

(2 pts) \( = \frac{1}{2} \left[ \frac{4}{3} \pi \theta + 4 \sin \theta + 2 \theta + \sin(2\theta) \right] \)

\( = \frac{1}{2} \left[ (4 \pi - 4 \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} - 2 \pi - 4 \frac{\sqrt{3}}{2} - (-\frac{\sqrt{3}}{2})) \right] \)

\( = \frac{1}{2} (2\pi + (2 - 8) \frac{\sqrt{3}}{2}) \)

(2 pts) \( = \pi - \frac{3}{2} \sqrt{3} \)

2-3. Given the surface with the following equation in rectangular coordinates:

\( z^2 = 2x^2 + 2y^2, z \geq 0. \)

2. Write this equation in cylindrical coordinates.

(3 pts) \( x = r \cos \theta, y = r \sin \theta, z = z \)

(2 pts) \( z^2 = 2r^2 \cos^2 \theta + 2r^2 \sin \theta \)

(2 pts) \( z^2 = 2r^2 (\cos^2 \theta + \sin^2 \theta) \)

(2 pts) \( z^2 = 2r^2 \)

(1 pt) \( z = \sqrt{z} r \ (r > 0) \)
3. Write this equation in spherical coordinates.
   (3 pts) \( x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi \)

   \[ \rho^2 \cos^2 \phi = 2 \rho^2 \sin^2 \phi \cos^2 \theta + 2 \rho^2 \sin^2 \phi \sin^2 \theta \]
   (2 pts) \( \cos^2 \phi = 2 \sin^2 \phi \cos^2 \theta + 2 \sin^2 \phi \sin^2 \theta \)
   \( \cos^2 \phi = 2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) \)
   (2 pts) \( \cos^2 \phi = 2 \sin^2 \phi \)

   (2 pts) \( \frac{1}{2} = \tan^2 \phi \rightarrow \tan \phi = \sqrt{\frac{1}{2}} \)

   (1 pt) \( \phi = \tan^{-1} \sqrt{\frac{1}{2}} \)

4-5. Given the curve: \( \vec{r}(t) = < \cos(2t), \sin(t), t^2 > \),

4. Find an equation of the tangent line at the point \( (0, \frac{\sqrt{15}}{2}, \frac{\pi^2}{16}) \).

   \( \vec{r}'(t) = < \cos(2t), \sin(t), 2t > \rightarrow <0, \frac{\sqrt{15}}{2}, \frac{\pi^2}{16}> \)

   (3 pts) \( \cos(2t) = 0, \sin(t) = \frac{\sqrt{15}}{2}, t^2 = \frac{\pi^2}{16} \rightarrow t = \frac{\pi}{4} \) (we rule out \( t = -\frac{\pi}{4} \) by \( \sin(t) = \frac{\sqrt{15}}{2} \))

   (2 pts) \( \vec{r}'(t) = <-2 \sin(2t), \cos(t), 2t > ; \)

   (2 pts) \( \vec{r}'\left( \frac{\pi}{4} \right) = <-2 \sin(\frac{\pi}{4}), \cos(\frac{\pi}{4}), 2 \frac{\pi}{4} > = <-2, \frac{\sqrt{15}}{2}, \frac{\pi}{2} > \)

   (3 pts) Line equation: \( \vec{v}(s) = <0, \frac{\sqrt{15}}{2}, \frac{\pi^2}{16}> + s <2, \frac{-\sqrt{15}}{2}, \frac{\pi}{2} > \)

5. Find the unit tangent vector \( \vec{T} \) at \( t = \frac{\pi}{2} \).

   (3 pts) \( \vec{T} = \frac{\vec{r}'(t)}{||\vec{r}'(t)||} \)

   (3 pts) \( \frac{< -2 \sin(2t), \cos(t), 2t >}{||< -2 \sin(2t), \cos(t), 2t >||} = \frac{< -2 \sin(2t), \cos(t), 2t >}{\sqrt{4 \sin^2(2t) + \cos^2(t) + 4t^2}} \)

   (3 pts) \( \vec{T}\left( \frac{\pi}{2} \right) = \frac{<0, 0, \pi>}{\sqrt{\pi^2}} \)

   (1 pt) \( = <0, 0, 1> \)

6. \( \int_0^{\ln(2)} e^t \vec{i} + t^2 \vec{j} + \cos(t) \vec{k} dt = \)

   (3 pts) \( = \int_0^{\ln(2)} e^t dt \vec{i} + \int_0^{\ln(2)} t^2 dt \vec{j} + \int_0^{\ln(2)} \cos(t) dt \vec{k} \)

   Note: you can use the calculator here provided you get it right; to get further partial credit you must show work as shown here:

   (3 pts) \( (e^{\ln(2)} - e^0) \vec{i} + \frac{1}{4} (\ln(2))^3 \vec{j} + (\sin(\ln(2)) - \sin(0)) \vec{k} \)

   (4 pts) \( \vec{k} + \frac{1}{4} (\ln(2))^3 \vec{j} + \sin(\ln(2)) \vec{k} \)
This problem is for the 9 am section only.

7. If \( \mathbf{r}(t) = \langle \sqrt{2} t, e^{-t}, e^t \rangle, t \in [0, 1], \) compute the length of the curve.

(1 pt) \( \mathbf{r}'(t) = \langle \sqrt{2}, -e^{-t}, e^t \rangle \)

(1 pt) \( |\mathbf{r}'(t)| = \sqrt{2 + e^{-2t} + e^{2t}} \)

(3 pts) \( L = \int_0^1 \sqrt{2 + e^{-2t} + e^{2t}} \, dt = \)

(again, you can use the calculator, but if you want partial credit you need to show work as follows)

(2 pts) \( \int_0^1 \sqrt{(e^{-t} + e^t)^2} \, dt = \)

(2 pts) \( \int_0^1 (e^{-t} + e^t) \, dt = \int_0^1 (e^{-t} + e^t) \, dt \) (note: \( e^{-t} + e^t > 0 \))

(1 pt) \( e^{-1} + e - 2 \)