MTH 223 Quiz Three.
Take Home. Be neat and show your work.

Recall that $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$ and $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$

1. Show that $\cosh^2(x) - \sinh^2(x) = 1$

$$\cosh^2(x) - \sinh^2(x) = \frac{1}{4}(e^{2x} + 2e^xe^{-x} + e^{-2x}) - \frac{1}{4}(e^{2x} - 2e^xe^{-x} + e^{-2x}) = \frac{1}{4}(2e^xe^{-x} - (-2e^xe^{-x})) = \frac{1}{4}(2e^0 - (-2e^0)) = \frac{1}{4}(4) = 1$$

2. Show that $\frac{d}{dx} \cosh(x) = \sinh(x)$ and that $\frac{d}{dx} \sinh(x) = \cosh(x)$

$$\frac{d}{dx} \cosh(x) = \frac{d}{dx} \left( \frac{1}{2} e^x + \frac{1}{2} e^{-x} \right) = \frac{1}{2} e^x - \frac{1}{2} e^{-x} = \sinh(x)$$

$$\frac{d}{dx} \sinh(x) = \frac{d}{dx} \left( \frac{1}{2} e^x - \frac{1}{2} e^{-x} \right) = \frac{1}{2} e^x + \frac{1}{2} e^{-x} = \cosh(x)$$

Now let $\vec{r}'(t) = \langle \cosh(t), \sinh(t), t \rangle$

3. Find $\vec{v}(t) = \langle \sinh(t), \cosh(t), 1 \rangle$

4. Find $\vec{a}(t). = \langle \cosh(t), \sinh(t), 0 \rangle$

5. Find $v(t) = \|\vec{r}'(t)\| = \sqrt{\cosh^2 t + \sinh^2(t) + 1} = \sqrt{2 \cosh^2 t}$ or $\sqrt{2 + 2\sinh^2 t} = \sqrt{2} \cosh(t)$

6. Find the arclength of $\vec{r}'(t)$ from $t = 0$ to $t = 1$. **No decimal answers, please!**

$$\int_0^1 \sqrt{\cosh^2 t + \sinh^2 t + 1} \, dt = \int_0^1 \sqrt{2} \cosh t \, dt = \int_0^1 \sqrt{2} |\cosh(t)| \, dt = \int_0^1 \sqrt{2} \cosh(t) \, dt = \sqrt{2} \big|_0^1 \sinh(t) = \sqrt{2} \left( e - \frac{1}{e} \right).$$

7. Find $\vec{T} = \frac{\cosh(t), \sinh(t), 1}{\sqrt{2} \cosh(t)} = \frac{1}{\sqrt{2}} \langle \cosh(t), \sinh(t), 1, \frac{1}{\cosh(t)} \rangle$

which is sometimes written as $\frac{1}{\sqrt{2}} \langle \tanh(t), 1, \sec(h(t) \rangle$

8. Find $\alpha_T = \frac{d}{dt} v = \frac{d}{dt} \sqrt{2} \cosh(t) = \sqrt{2} \sinh(t)$.

Or, you might have written it, as say,

$$\frac{1}{2} \left( \cosh^2 t + \sinh^2 t + 1 \right)^{-\frac{3}{2}} (2 \cosh(t) \sinh(t) + 2 \cosh(t) \sinh(t))$$

$$= \frac{1}{2} \frac{4 \cosh(t) \sinh(t)}{\sqrt{\cosh^2 t + \sinh^2 t + 1}}$$

which equals $2 \frac{\cosh(t) \sinh(t)}{\sqrt{2} \cosh(t)} = \sqrt{2} \sinh(t)$
9. When $t = 1$ find $\kappa$ (curvature) $\kappa = \frac{|\gamma''(t)|}{|\gamma'(t)|^3}$

I know I told you to plug in for $t$ but I'll go against that advice to demonstrate something:

$\gamma'(t) = \langle \sinh(t), \cosh(t), 1 \rangle$, $\gamma''(t) = \langle \cosh(t), \sinh(t), 0 \rangle$

$\gamma' \times \gamma'' = \langle -\sinh(t), \cosh(t), \sinh^2 t - \cosh^2 t \rangle = \langle -\sinh(t), \cosh(t), -1 \rangle$

$\|\gamma' \times \gamma''\| = \sqrt{\sinh^2(t) + \cosh^2(t) + 1} = \sqrt{\cosh^2 1 - 1 + \cosh^2 t + 1}$

$\kappa = \frac{\sqrt{2} \cosh(t)}{\sqrt{2} \cosh^3(t)} = \frac{1}{2 \cosh^2 t}$

Notice that as $t$ gets large in magnitude, curvature gets small. Maximum curvature is at $t = 0$

At $t = 1$, it is $\frac{1}{2 \cosh^2(1)} = \frac{1}{2 \left( \frac{1}{e} + \frac{1}{e} \right)^2} = \frac{2}{e^2 + 2 + \frac{1}{e^2}} = \frac{2e^2}{e^4 + 2e^2 + 1}$

10. When $t = 1$ find $a_N$

Use $a_N = \kappa \nu^2$ $a_N = \frac{1}{2 \cosh^2 t} \ast (\sqrt{2} \cosh(t))^2 = 1$

You might think about what this means.

11. At $t = 1$ find an equation of the tangent line.

The equation is $\gamma(1) + u \gamma'(1) = \langle \cosh(1), \sinh(1), 1 \rangle + u \langle \sinh(1), \cosh(1), 0 \rangle$

OR: $x(u) = \cosh(1) + u \sinh(1)$, $y(u) = \sinh(1) + u \cosh(1)$, $z(u) = 1 + u$

12. At $t = 1$ find an equation of the normal plane.

$< x - \cosh(1), y - \sinh(1), z - 1 > \cdot < \sinh(1), \cosh(1), 1 > = 0$

$\sinh(1)(x - \cosh(1)) + \cosh(1)(y - \sinh(1) + z - 1) = 0$