M 223 quiz.

For both 1 and 2: does the following limit exist? If not, back up your answer. If so, say what the limit is and give a brief reason why:

1. \( \lim_{(x,y) \to (0,0)} \frac{x^2 + \cos(xy)}{x^2 + y^2 + 1} \)
   Solution: note that \( f \) is continuous at \( (0,0) \) so the limit exists and is equal to \( \frac{0^2 + \cos(0)}{0^2 + 0^2 + 1} = 1 \).

2. \( \lim_{(x,y) \to (0,0)} \frac{2xy}{x^2 + y^2} \)
   Solution: approach the origin along the line \( y = mx \); get:
   \[
   \lim_{(x,mx) \to (0,0)} \frac{2xmx}{x^2 + (mx)^2} = \lim_{(x,mx) \to (0,0)} \frac{2x^2m}{x^2 + m^2x^2} = \lim_{(x,mx) \to (0,0)} \frac{2m}{1 + m^2} \text{ which varies by } m.
   \]
   Therefore the limit does not exist.

3. Calculate \( \nabla f \) if \( f = 2x^2e^y + \sin(x + y) \)
   \( \Rightarrow f_x, f_y \Rightarrow 4xe^y + \cos(x + y), 2x^2e^y + \cos(x + y) > \)

4-5: \( f(x,y) = x^3y + \ln(x-y) \)

4. Find the directional derivative of at \( (2,1) \) in the direction of \( < -1,2 > \).
   \( \nabla f = < 3x^2y + \frac{1}{x-y}, x^3 - \frac{1}{x-y} >, \nabla f(2,1) = < 12 + 1, 8 - 1 > = < 13,7 > \)
   unit vector in the direction of \( < -1,2 > \) is \( \frac{1}{\sqrt{5}} < -1,2 > = \frac{1}{\sqrt{5}} < -1,2 > \)
   \( \nabla f(2,1) \cdot \frac{1}{\sqrt{5}} < -1,2 > = \frac{1}{\sqrt{5}} (-13 + 14) = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5} \)

5. In what direction is the rate of change of \( f \) at \( (2,1) \) the greatest? What is that rate of change?
   The direction is in the direction of \( < 13,7 > \) which is the gradient vector and the rate of change is:
   \( \sqrt{13^2 + 7^2} = 218 \)