M223 Self-scored Quiz.
Use any valid technique.

1. \( \int_C \vec{F} \cdot d\vec{r} \) where \( C \) is the square with vertices \((0,0), (1,0), (1,1)\) and \((0,1)\) taken in the counterclockwise direction and \( \vec{F} = x^2y\hat{i} + e^x\hat{j} \)

Here it would be a good idea to use Green’s theorem (so we can merely integrate over a rectangle)
\[
\frac{\partial Q}{\partial x} = e^x, \quad \frac{\partial P}{\partial y} = x^2 \rightarrow \int_C Pdx + Qdy = \iint_{\Omega} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \int_0^1 \int_0^2 e^x - x^2 dy dx = \int_0^1 \left[ e^x - \frac{1}{2}x^3 \right] = e - \frac{1}{2} - 1 = e - \frac{4}{3}
\]

2. \( \int_C \vec{F} \cdot d\vec{r} \) where \( C \) is the circle \( x^2 + y^2 = 1 \) taken in the counterclockwise direction and \( \vec{F} = < \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2} > \)

Here we CAN’T use Green’s theorem as \( \vec{F} \) is not smooth at \((0,0)\) and \( C \) encloses \((0,0)\).

So we compute \( \int_C \vec{F} \cdot d\vec{r} = \int \frac{y}{x^2+y^2} dx - \frac{x}{x^2+y^2} dy \) directly. Let \( x = \cos(t), y = \sin(t) \)

Then \( 0 \leq t \leq 2\pi \) and \( dx = -\sin(t) dt, dy = \cos(t) dt \) and \( x^2 + y^2 = 1 \)

So:
\[
\int_C \frac{y}{x^2+y^2} dx - \frac{x}{x^2+y^2} dy = \int_C \frac{\sin(t)}{1} (-\sin(t)) - \frac{\cos(t)}{1} \cos(t) dt = \int_0^{2\pi} -\sin^2 t - \cos^2 t dt = -\int_0^{2\pi} 1 dt = -2\pi
\]

3. \( \int_C \vec{F} \cdot d\vec{r} \) where \( C \) is the curve that runs from \((0,0)\) to \((1,1)\) along \( x = \sqrt{y} \) and then back to \((0,0)\) along the path \( x = y^2 \) and \( \vec{F} = < 2x\sin y - e^x, x^2 \cos(y) + \tan(y) > \)

This looks messy. Green’s Theorem applies and so we get
\[
\int_C \vec{F} \cdot d\vec{r} = \int_C Pdx + Qdy = \iint_{\Omega} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA, \quad \frac{\partial Q}{\partial x} = 2x \cos(y), \quad \frac{\partial P}{\partial y} = 2x \cos(y)
\]

So
\[
\int \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = \int \int_{\Omega} 0 \, dA = 0.
\]

Note: \( \vec{F} \) is a conservative vector field and Green’s theorem will always pick that up (in the plane).

4. \( \int_C f(x,y)ds \) where \( f(x,y) = x + y^2 \) and \( C \) is the path that consists of line segments running from \((0,0)\) to \((1,0)\) and then to \((1,1)\).

We need to take this in two segments (this is an integral of a scalar function, NOT a vector field integral)
Segment 1: \((0, 0)\) to \((1, 0)\) has \(x(t) = t, y(t) = 0, x'(t) = 1, y'(t) = 0 \rightarrow ds = \sqrt{1^2 + 0^2}, 0 \leq t \leq 1 \)
\[
\int_{c_1} x + y^2 ds = \int_0^1 (t + 0^2) \sqrt{1 + 0^2} \, dt = \int_0^1 t \, dt = \frac{1}{2}
\]
Segment 2: \((1, 0)\) to \((1, 1)\) \(x(t) = 1, y(t) = t, x'(t) = 0, y'(t) = 1 \rightarrow ds = \sqrt{0^2 + 1^2} \)
\[
\int_{c_2} x + y^2 ds = \int_0^1 (1 + t^2) \sqrt{1 + 0^2} \, dt = \left| \int_0^1 t \, dt + \frac{1}{3} t^3 \right| = \frac{4}{3}
\]
5. \(\int_C (y^2 + 2xy)dx + (y^2x)dy\) where \(C\) is the circle \((x - 2)^2 + (y + 3)^2 = 4\) taken in the standard counterclockwise direction.

Green's Theorem applies and is by far the easier way. \(\int_C Pdx + Qdy = \int \int_\Omega \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA\)

So \(\frac{\partial Q}{\partial x} = y^2, \frac{\partial P}{\partial y} = x^2 - 2x \rightarrow \int_C (yx^2 + 2xy)dx + (y^2x)dy = \int \int_\Omega y^2 - x^2 + 2xdA\)

Now to handle our region, we switch to modified polar coordinates:
\(x - 2 = 2r \cos(\theta), y + 3 = 2r \sin(\theta) \rightarrow x = 2r \cos \theta + 2, y = 2r \sin \theta - 3\)

\[J = \left| \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \right| = \left| \det \begin{bmatrix} 2 \cos \theta & -2r \sin \theta \\ 2 \sin \theta & 2r \cos \theta \end{bmatrix} \right| = |4r \cos^2 \theta + 4r \sin^2 \theta| = 4r\]

\[\int \int_\Omega y^2 - x^2 + 2xdA = \int_0^{2\pi} \int_0^2 (2r \sin \theta - 3)^2 - (2r \cos \theta + 2)^2 + 2(2r \cos \theta + 2))4r \, dr \, d\theta\]

\[\int_0^{2\pi} \int_0^2 (4r^2 \sin^2 \theta - 12r \sin \theta + 9 - (4r^2 \cos^2 \theta + 8r \cos \theta + 4) + 4r \cos \theta + 4))4r \, dr \, d\theta = \]

\[4 \int_0^{2\pi} \int_0^2 4r^3(\sin^2 \theta - \cos^2 \theta) - 12r^2 \sin \theta - 4r^2 \cos \theta + 9r \, dr \, d\theta = \]

\[4 \int_0^{2\pi} \int_0^2 4r^3(\sin^2 \theta - \cos^2 \theta) - 4r^3 \sin \theta - \frac{4}{3} r^3 \cos \theta + \frac{9}{2} r^2 \, dr \, d\theta = \]

\[4 \int_0^{2\pi} 16(\sin^2 \theta - \cos^2 \theta) - 32 \sin \theta - \frac{32}{3} \cos \theta + 18 \, d\theta = \]

\[4 \int_0^{2\pi} 16(\cos(2\theta)) - 32 \sin \theta - \frac{32}{3} \cos \theta + 18 \, d\theta = 4 \cdot 2\pi \cdot 18 = 144\pi \]